Tensor products of quiver bundles

Overview

In this work we study a notion of **tensor product** for **quiver representations** in the category of **$\mathcal{O}_X$-modules**. We study the **polystability** of this representations and use this to deduce a quiver analogue of the classical **Segre embedding**. Also, we use tensorization of representations two identify distinguished closed subschemes of $\mathrm{GL}(n)$-**character varieties** of free abelian groups.

The setup

Let $Q = (V, E, h, t)$ be a quiver and $(X, \mathcal{O}_X)$ be a ringed space.

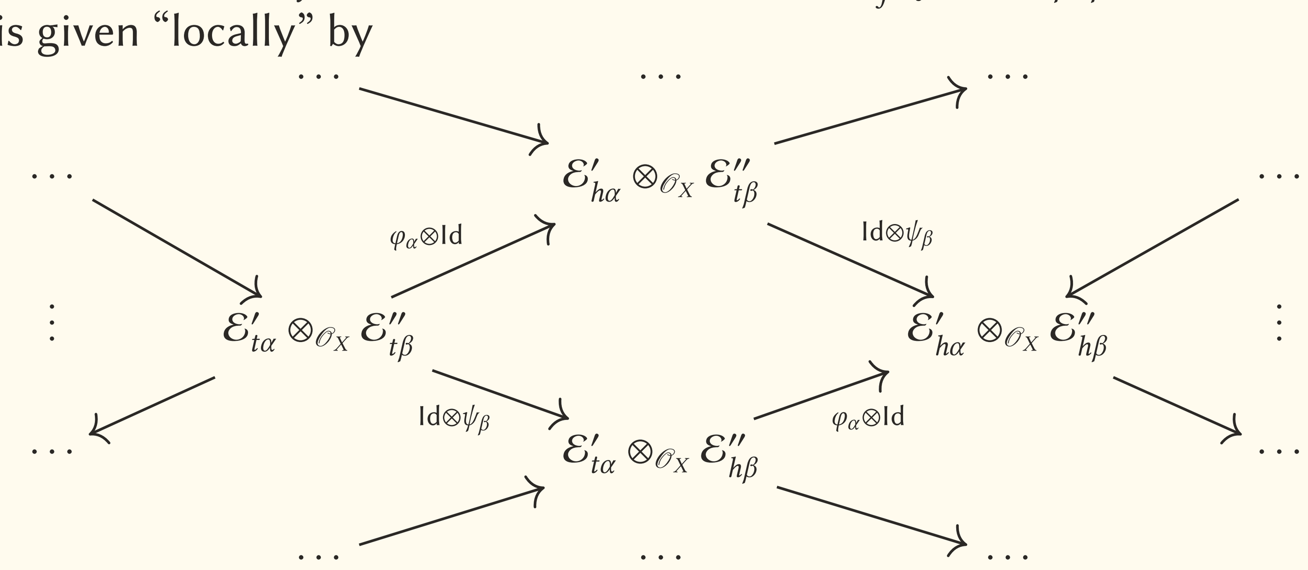
- A **representation** of Q is a tuple
$$\mathcal{E} = ((\mathcal{E}_i)_{i\in V}, (\varphi_\alpha : \mathcal{E}_{t\alpha} \to \mathcal{E}_{h\alpha})_{\alpha\in E}).$$
- The **path algebra** of Q is the sheaf of $\mathcal{O}_X$-algebras
$$\mathcal{A}_Q = \bigoplus_{p \text{ path}} \mathcal{O}_X$$
with product rule given by
$$s_p \cdot s_q = \begin{cases} s_p s_q & \text{if } hq = tp, \\ 0 & \text{otherwise.} \end{cases}$$
- Let $\mathcal{I}$ be an ideal sheaf. We call an $\mathcal{O}_X$-module $\mathcal{M}$ together with an $\mathcal{O}_X$-linear map
$$\mu : \mathcal{A}_Q/\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M},$$
for which the usual axioms of modules over algebras hold, a **$\mathcal{A}_Q/\mathcal{I}$-module**.
- **Theorem:** The categories $\mathrm{Rep}(\mathcal{A}_Q)$ and $\mathcal{A}_Q$-mod are equivalent.

Tensor product of quiver representations

Two main ingredients:

- Let Q', Q'' be quivers. The **tensor product quiver** $Q' \otimes Q''$ is the quiver given by the following data: $V = V' \times V''$, $E = (V' \times E'') \sqcup (E' \times V'')$, $h(\alpha, j) = (h\alpha, j)$ and so on.
- There is an isomorphism $\mathcal{A}_{Q'\otimes Q''}/\mathcal{I} \cong \mathcal{A}'_Q \otimes_{\mathcal{O}_X} \mathcal{A}_{Q''}$ where $\mathcal{I}$ is the ideal sheaf of relations generated by
$$(h\alpha, \beta)(\alpha, t\beta) - (\alpha, h\beta)(t\alpha, \beta),\ \alpha \in E',\ \beta \in E''.$$

Let $\mathcal{E}' \in \mathrm{Rep}(Q')$ and $\mathcal{E}'' \in \mathrm{Rep}(Q'') \Longrightarrow$ We have $\mathcal{M}_{\mathcal{E}'} \in \mathcal{A}_{Q'}$-mod and $\mathcal{M}_{\mathcal{E}''} \in \mathcal{A}_{Q''}$-mod $\Longrightarrow$ The tensor product $\mathcal{M}_{\mathcal{E}'} \otimes_{\mathcal{O}_X} \mathcal{M}_{\mathcal{E}''}$ is a $\mathcal{A}_{Q'} \otimes_{\mathcal{O}_X} \mathcal{A}_{Q''}$ module $\Longrightarrow$ The tensor product $\mathcal{M}_{\mathcal{E}'} \otimes_{\mathcal{O}_X} \mathcal{M}_{\mathcal{E}''}$ is a $\mathcal{A}_{Q'\otimes Q''}/\mathcal{I}$ module $\Longrightarrow$ This $\mathcal{A}_{Q'\otimes Q''}/\mathcal{I}$ module corresponds to a representation of $Q' \otimes Q''$ with relations which we call the **tensor product** $\mathcal{E}' \otimes \mathcal{E}''$. Suppose that $\mathcal{E}' = ((\mathcal{E}'_i)_{i\in V'}, (\varphi_\alpha)_{\alpha\in E'})$ and $\mathcal{E}'' = ((\mathcal{E}''_j)_{j\in V''}, (\psi_\beta)_{\beta\in E''})$, then its tensor product is given "locally" by



Remark:

- When $X = \mathrm{Spec}(\mathbb{C})$ we recover the notion of tensor product of representations studied by Herschend [Her08] and Das et al. [DDR24].
- The above construction also works for **twisted quiver representations** and **representations with relations**.

Polystability of quiver bundles

From now on we assume X = Compact Riemann surface/Point and we consider quiver representations in the category of Holomorphic vector bundles over X/Finite dimensional $\mathbb{C}$-vector spaces.

- Let $\mathcal{E} = ((\mathcal{E}_i)_{i\in V}, (\varphi_\alpha)_{\alpha\in E})$ be a representation of Q. For $\theta = (\theta_i)_{i\in V} \in \mathbb{R}^{|V|}$ we define the θ-**slope** of $\mathcal{E}$ to be
$$\mu_\theta(\mathcal{E}) = \frac{\sum_{i\in V} \deg(\mathcal{E}_i) + \theta_i \mathrm{rk}(\mathcal{E}_i)}{\sum_{i\in V} \mathrm{rk}(\mathcal{E}_i)}.$$
- $\mathcal{E}$ is said to be θ-**(semi)stable** if for all non-trivial subrepresentation $\mathcal{F}$ we have
$$\mu_\theta(\mathcal{F})(\leq) < \mu_\theta(\mathcal{E}).$$
- We say that $\mathcal{E}$ is θ-**polystable** if it is a direct sum of θ-stable representations of the same θ-slope.

Theorem:[ACGP03] A quiver bundle $\mathcal{E} = ((\mathcal{E}_i)_{i\in V}, (\varphi_\alpha)_{\alpha\in E})$ is θ-polystable if and only if there exist an hermitian metric H_i on each $\mathcal{E}_i$ such that
$$\sqrt{-1}\Lambda F_i + (\sum_{h\alpha=i} \varphi_\alpha \varphi_\alpha^* - \sum_{t\alpha=i} \varphi_\alpha^* \varphi_\alpha) = \theta_i \mathrm{Id}_{\mathcal{E}_i},\ \forall i \in V.$$

- $F_i :=$ Curvature of the Chern connection associated to H_i.
- $\Lambda : \Omega^{(i,j)}(X) \to \Omega^{(i-1,j-1)}(X)$ contraction operator w.r.t. a fixed Kähler form on X.

Theorem: Let $\mathcal{E}', \mathcal{E}''$ be θ' and θ''-polystable quiver bundles respectively. Then, $\mathcal{E}' \otimes \mathcal{E}''$ is θ-polystable for $\theta = (\theta'_i + \theta''_j)_{(i,j)\in V'\times V''}$.

Segre embedding for quiver representations

Let $\mathcal{M}^{\theta-ss}(Q, d)$ be the moduli space of θ-semistable quiver representations of dimension $d \in \mathbb{N}^{|V|}$.

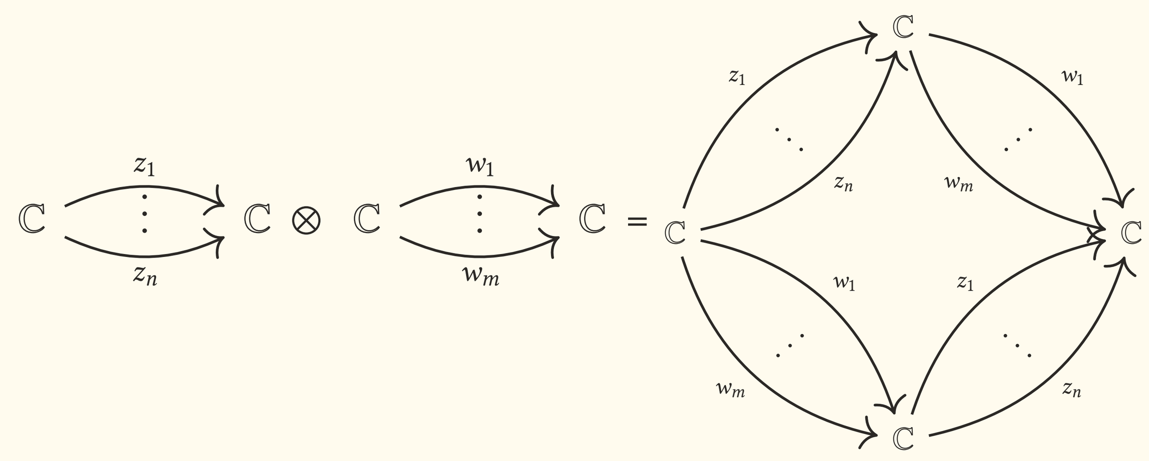
- Two points of view on the moduli space:
$$\mathcal{M}^{\theta-ss}(Q, d) = \mathrm{Rep}(Q, d) /\!\!/_\theta \mathrm{GL}(d) \simeq \mu^{-1}(\theta)/\mathrm{U}(d)$$
where
$$\mu^{-1}(\theta) = \{\varphi \in \mathrm{Rep}(Q, d) | \sum_{h\alpha=i} \varphi_\alpha \varphi_\alpha^* - \sum_{t\alpha=i} \varphi_\alpha^* \varphi_\alpha = \theta_i \mathrm{Id}\}.$$
- The points of the moduli space classify S-equivalence classes of semistable representations or, equivalently, polystable representations modulo isomorphism.

Theorem: Let Q', Q'' be quivers, $d' \in \mathbb{N}^{|V'|}$, $d'' \in \mathbb{N}^{|V''|}$, $\theta' \in \mathbb{R}^{|V'|}$, $\theta'' \in \mathbb{R}^{|V''|}$ be generic dimension vectors and stability parameters respectively so that the corresponding symplectic reductions are manifolds. Then, the tensorization map
$$\begin{array}{ccc} \mathcal{M}^{\theta'-s}(Q', d') \times \mathcal{M}^{\theta''-s}(Q'', d'') & \longrightarrow & \mathcal{M}^{\theta-s}(Q' \otimes Q'', d) \\ ([\varphi = (\varphi_\alpha)_{\alpha\in E'}], [\psi = (\psi_\beta)_{\beta\in E''}]) & \longmapsto & [(\varphi_\alpha \otimes \mathrm{Id})_{(\alpha,j)\in E'\times V''} \sqcup (\mathrm{Id} \otimes \psi_\beta)_{(i,\beta)\in V'\otimes E''}] \end{array}$$
is an embedding for $\theta = (\theta'_i + \theta''_j)_{(i,j)\in V'\times V''}$ and $d = (d'_i d''_j)_{(i,j)\in V'\times V''}$. The image is then a submanifold of real dimension $-2(\langle d', d'\rangle_{Q'} + \langle d'', d''\rangle_{Q''})$.

Remark:

- The bilinear form on $\mathbb{Z}^{|V|}$,
$$\langle d, d'\rangle_Q := \sum_{i\in V} d_i d'_i - \sum_{\alpha\in E} d_{t\alpha} d'_{h\alpha}$$
is the so-called **Euler form** associated to Q.
- The tensorization map can be seen to be algebraic under certain conditions of the quivers and the stability parameters.

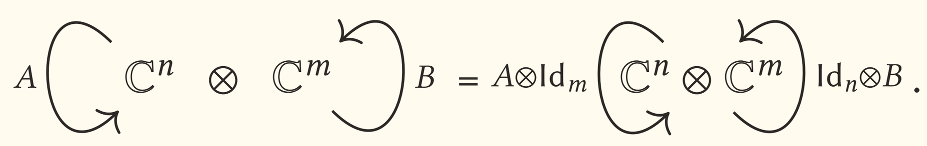
Example: If we apply the previous theorem to the tensorization of representations below we recover the closed subvariety that the **classical Segre embedding** describes.



Tensorization and character varieties

Figure: The Jordan quiver Q_J and the tensor quiver $Q := Q_J \otimes Q_J$

Lemma: Let $\mathcal{R} = \{\alpha\beta - \beta\alpha\}$ and $\mathcal{M}(Q, d, \mathcal{R}) = \mathrm{Rep}(Q, d, \mathcal{R}) /\!\!/ \mathrm{GL}(d)$. There is a map
$$\mathcal{M}(Q_J, n) \times \mathcal{M}(Q_J, m) \longrightarrow \mathcal{M}(Q, d, \mathcal{R})$$
of affine schemes induced by tensorization of representations:



The $\mathrm{GL}(n)$ **character variety** of the group $\mathbb{Z}^r$ is the GIT quotient:
$$\mathcal{M}_{r,n} := \mathrm{Hom}(\mathbb{Z}^r, \mathrm{GL}(n)) /\!\!/ \mathrm{GL}(n) = \{(A_1, \ldots, A_r) \in \mathrm{GL}(n)^r \,|\, [A_k, A_j] = 0\} /\!\!/ \mathrm{GL}(n).$$

- $\mathcal{M}_{1,n} \overset{\text{open affine}}{\hookrightarrow} \mathcal{M}(Q_J, n)$ and $\mathcal{M}_{2,mn} \overset{\text{open affine}}{\hookrightarrow} \mathcal{M}(Q, mn, \mathcal{R})$ so the tensorization map restricts to a morphism
$$\mathcal{M}_{1,n} \times \mathcal{M}_{1,m} \longrightarrow \mathcal{M}_{2,mn}$$
of affine schemes.
- The scheme-theoretic image is the affine closed subscheme of the character variety $\mathcal{M}_{2,mn}$ determined by the kernel of the ring map
$$\mathbb{C}[\mathcal{M}_{2,mn}] \to \mathbb{C}[\mathcal{M}_{1,n}] \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{M}_{1,m}]$$
associated to the morphism above.
- As a topological space, the scheme-theoretic image is the closure of the set-theoretic one. In this way we recover an irreducible ($\mathcal{M}_{1,n} \times \mathcal{M}_{1,m}$ is irreducible) closed subscheme of $\mathcal{M}_{2,mn}$ which we call the **Segre subscheme**.
- A similar strategy can be used to obtain Segre subschemes for higher rank character varieties from morphisms
$$\mathcal{M}_{1,n_1} \times \cdots \times \mathcal{M}_{1,n_k} \to \mathcal{M}_{k,n_1\ldots n_k}$$
induced by tensorization of representations.

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