Tensor products of Quiver bundles

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Motivation

- Quiver bundles arise as fixed points for the natural $\mathbb{C}^*$-action on the moduli space of Higgs bundles over a compact Riemann surface.
- Gothen and Nozad [GN19] introduced a quiver bundle whose polystability has important consequences on the deformation theory of the moduli spaces of Holomorphic chains. This turns out to be an instance of the notion of tensor product of quiver representations introduced in this work.

Agenda

1. Basic definitions and notation

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2. Representation theory of tensor quivers

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3. Moduli of quiver representations and the quiver vortex equations

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3. Moduli of quiver representations and the quiver vortex equations
4. Applications

Basic Definitions

Quivers

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- A path is a sequence of edges

$$p := \alpha_1 \cdots \alpha_k$$

such that $h\alpha_j = t\alpha_{j+1}$. We denote the trivial paths e_i for all $i \in V$.

Twisted representations

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- A $\mathscr{M}$-twisted representation of Q is a tuple $\mathcal{E} = ((\mathcal{E}_i)_{i \in V}, (\varphi_\alpha : \mathscr{M}_\alpha \otimes_{\mathcal{O}_X} \mathcal{E}_{t\alpha} \to \mathcal{E}_{h\alpha})_{\alpha \in E})$.

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- A morphism $f : \mathcal{E} \to \mathcal{E}'$ of twisted representations is given by a family $f = (f_i : \mathcal{E}_i \to \mathcal{E}'_i)_{i \in V}$ for which the diagram

$$\begin{tikzcd} \mathscr{M}_\alpha \otimes_{\mathcal{O}_X} \mathcal{E}_{t\alpha} \arrow[r, "\varphi_\alpha"] \arrow[d, "\mathrm{Id} \otimes f_{t\alpha}"'] & \mathcal{E}_{h\alpha} \arrow[d, "f_{h\alpha}"] \\ \mathscr{M}_\alpha \otimes_{\mathcal{O}_X} \mathcal{E}'_{t\alpha} \arrow[r, "\varphi'_\alpha"'] & \mathcal{E}'_{h\alpha} \end{tikzcd}$$

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Remark
We can work instead with the full subcategories of representations in the categories of coherent sheaves over a Kähler manifold or (quasi-)coherent sheaves over a scheme.

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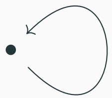
One can think of such an object as a K_X^*-twisted quiver representation of the so-called Jordan quiver, Q_J.



The Jordan quiver Q_J

Twisted path algebra of a quiver

- For $p = \alpha_1 \cdots \alpha_k$ a non-trivial path of Q, we define

$$\mathscr{M}_p = \mathscr{M}_{\alpha_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathscr{M}_{\alpha_k}$$

  and for $p = e_i$, we set

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- The twisted path algebra of Q is the $\mathcal{O}_X$-module

$$\mathcal{T}_{\mathscr{M}}\mathcal{A}_Q = \bigoplus_{p \text{ path}} \mathscr{M}_p$$

  with product rule given by

$$m_p \cdot m_q = \begin{cases} m_p \otimes m_q & \text{if } hq = tp, \\ 0 & \text{otherwise.} \end{cases}$$

Modules over the path algebra

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- $\mathcal{M}$ is determined by an $\mathcal{O}_X$-linear map

$$\mu : \mathcal{T}_{\mathscr{M}}\mathcal{A}_Q \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$$

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- Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{T}_{\mathscr{M}}\mathcal{A}_Q$-mod. A morphism $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a $\mathcal{O}_X$-linear map such that the diagram

$$\begin{tikzcd}
\mathcal{T}_{\mathscr{M}}\mathcal{A}_Q \otimes_{\mathcal{O}_X} \mathcal{M}_1 \arrow[r, "\mu_1"] \arrow[d, "\mathrm{Id}\otimes f"'] & \mathcal{M}_1 \arrow[d, "f"] \\
\mathcal{T}_{\mathscr{M}}\mathcal{A}_Q \otimes_{\mathcal{O}_X} \mathcal{M}_2 \arrow[r, "\mu_2"'] & \mathcal{M}_2
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  commutes.
- We denote this category as $\mathcal{T}_{\mathscr{M}}\mathcal{A}_Q$-mod.

Modules vs. Representations

Proposition [BBR20]

$\mathcal{T}_{\mathscr{M}}\mathcal{A}_Q$-mod $\simeq \mathsf{Rep}(\mathscr{M}Q)$

Representations with relations

- A relation on Q is a formal sum of the form

$$\sum_{\substack{p \text{ path}, \ |p| \geq 2 \\ tp=i, \ hp=j}} c_p \cdot p, \ \ c_p \in \mathcal{O}_X(X)$$

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- Let $\mathcal{R}$ be a set of relations and let $\mathcal{I}_{\mathcal{R}}$ any double-sided ideal of $\mathcal{T}_{\mathscr{M}}\mathcal{A}_Q$ generated by a subset of

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- A twisted representation satisfies the relations in $\mathcal{R}$ if for all $r = \sum m_p$ generator of $\mathcal{I}_{\mathcal{R}}(U)$,

$$\sum \psi_{m_p} : \mathcal{E}_{tp}(U) \to \mathcal{E}_{hp}(U) = 0.$$

Modules vs. Representations revisited

Proposition

Restriction of scalars gives a full embedding of categories $\mathcal{T}_{\mathscr{M}}\mathcal{A}_Q/\mathcal{I}_{\mathcal{R}}$-mod $\hookrightarrow \mathcal{T}_{\mathscr{M}}\mathcal{A}_Q$-mod.

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Theorem

There is an equivalence of categories between $\mathrm{Rep}(\mathscr{M}Q, \mathcal{R})$ *and* $\mathcal{T}_{\mathscr{M}}\mathcal{A}_Q/\mathcal{I}_{\mathcal{R}}\text{-mod}$ *such that the diagram of functors*

$$\begin{tikzcd}
\mathrm{Rep}(\mathscr{M}Q, \mathcal{R}) \arrow[r] \arrow[d, hook] & \mathcal{T}_{\mathscr{M}}\mathcal{A}_Q/\mathcal{I}_{\mathcal{R}}\text{-mod} \arrow[d, hook] \\
\mathrm{Rep}(\mathscr{M}Q) \arrow[r, "\simeq"'] & \mathcal{T}_{\mathscr{M}}\mathcal{A}_Q\text{-mod}
\end{tikzcd}$$

commute.

Representations of tensor quivers

The tensor product quiver

- Let Q', Q'' be quivers. The tensor product quiver $Q' \otimes Q''$ is the quiver given by the following data: $V = V' \times V''$, $E = (V' \times E'') \sqcup (E' \times V''), h(\alpha, j) = (h\alpha, j)$ and so on.

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- For instance

$$\bullet_1 \xrightarrow{\alpha} \bullet_2 \quad \otimes \quad \bullet_1 \xrightarrow{\beta} \bullet_2 \quad = \quad \begin{tikzcd} & \bullet_{(2,1)} \arrow[dr, "{(2,\beta)}"] & \\ \bullet_{(1,1)} \arrow[ur, "{(\alpha,1)}"] \arrow[dr, "{(1,\beta)}"'] & & \bullet_{(2,2)} \\ & \bullet_{(1,2)} \arrow[ur, "{(\alpha,2)}"'] & \end{tikzcd}$$

Tensor products of path algebras

Proposition

There is an isomorphism of twisted path algebras

$$\mathcal{T}_{\mathscr{M}}\mathcal{A}_{Q'\otimes Q''}/\mathcal{I} \cong \mathcal{T}_{\mathscr{M}'}\mathcal{A}_{Q'} \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathscr{M}''}\mathcal{A}_{Q''}.$$

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- $\mathscr{M}$ is the family of twisting modules given by $\mathscr{M}_{(\alpha,j)} := \mathscr{M}'_\alpha$ and $\mathscr{M}_{(i,\beta)} := \mathscr{M}''_\beta$.
- $\mathcal{I}$ is the ideal of relations generated by all the differences

$$m_{(h\alpha,\beta)} \otimes m_{(\alpha,t\beta)} - m_{(\alpha,h\beta)} \otimes m_{(t\alpha,\beta)}$$

with

$$\begin{aligned} m_{(h\alpha,\beta)} = m_{(t\alpha,\beta)} \in \mathscr{M}_{(h\alpha,\beta)}(U) = \mathscr{M}_{(t\alpha,\beta)}(U) = \mathscr{M}''_\beta(U) \\ m_{(\alpha,h\beta)} = m_{(\alpha,t\beta)} \in \mathscr{M}_{(\alpha,h\beta)}(U) = \mathscr{M}_{(\alpha,t\beta)}(U) = \mathscr{M}'_\alpha(U) \end{aligned}$$

and $U \subseteq X$ open.

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$\Longrightarrow$ We have $\mathcal{M}_{\mathcal{E}'} \in \mathcal{T}_{\mathscr{M}'} \mathcal{A}_{Q'}$-mod and $\mathcal{M}_{\mathcal{E}''} \in \mathcal{T}_{\mathscr{M}''} \mathcal{A}_{Q''}$-mod.

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$\Longrightarrow$ The tensor product $\mathcal{M}_{\mathcal{E}'} \otimes_{\mathcal{O}_X} \mathcal{M}_{\mathcal{E}''}$ is a $\mathcal{T}_{\mathscr{M}'}\mathcal{A}_{Q'} \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathscr{M}''}\mathcal{A}_{Q''}$ module.

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$\Longrightarrow$ The tensor product $\mathcal{M}_{\mathcal{E}'} \otimes_{\mathcal{O}_X} \mathcal{M}_{\mathcal{E}''}$ is a $\mathcal{T}_{\mathscr{M}}\mathcal{A}_{Q' \otimes Q''}/\mathcal{I}$ module.

$\Longrightarrow$ This $\mathcal{T}_{\mathscr{M}}\mathcal{A}_{Q' \otimes Q''}/\mathcal{I}$ module corresponds to a representation of $Q' \otimes Q''$ with relations which we call the tensor product $\mathcal{E}' \otimes \mathcal{E}''$.

The untwisted case

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$$
\begin{tikzcd}
 & \cdots \arrow[dr] & \cdots & \cdots & \\
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\vdots & \mathcal{E}'_{t\alpha} \otimes_{\mathcal{O}_X} \mathcal{E}''_{t\beta} \arrow[ur, "\varphi_\alpha \otimes \mathrm{Id}"] \arrow[dl] \arrow[dr, "\mathrm{Id} \otimes \psi_\beta"] & & \mathcal{E}'_{h\alpha} \otimes_{\mathcal{O}_X} \mathcal{E}''_{h\beta} \arrow[dr] & \vdots \\
\cdots & & \mathcal{E}'_{t\alpha} \otimes_{\mathcal{O}_X} \mathcal{E}''_{h\beta} \arrow[ur, "\varphi_\alpha \otimes \mathrm{Id}"] \arrow[dr] & & \cdots \\
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 & & \cdots \arrow[urr] & & \cdots & & \cdots & &
\end{tikzcd}$$

Remark
When $X = pt$ we recover the notion of tensor product studied by Herschend [Her08] and Das et al. [DDR24].

Moduli of quiver representations and the quiver vortex equations

Stability conditions of representations

From now on we assume X = Compact Riemann surface/Point and we consider quiver representations in the category of Holomorphic vector bundles over X/Finite dimensional $\mathbb{C}$-vector spaces.

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- We say that $\mathcal{E}$ is θ-polystable if it is a direct sum of θ-stable representations of the same θ-slope.

The linear case

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- For $\theta \in \mathbb{Z}^{|V|}$ such that $\theta \cdot d = 0$, there exist GIT quotients

$$\begin{array}{ccc} \mathrm{Rep}^{\theta-s}(Q,d)/\mathrm{GL}(d) & \xhookrightarrow{\text{open}} & \mathrm{Rep}^{\theta-ss}(Q,d) /\!\!/_\theta \mathrm{GL}(d) \\ := & & := \\ \mathcal{M}^{\theta-s}(Q,d) & & \mathcal{M}^{\theta-ss}(Q,d) \end{array}$$

Symplectic point of view

- Let $\mathrm{U}(d) = \prod_{i \in V} \mathrm{U}(d_i)$ be the unitary group w.r.t the hermitian metric
$$H((\varphi_\alpha)_{\alpha \in E}, (\psi_\alpha)_{\alpha \in E}) = \sum_{\alpha \in E} \mathrm{Tr}(\varphi_\alpha \psi_\alpha^*).$$

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$$\begin{array}{rccc} \mu : & \mathrm{Rep}(Q, d) & \longrightarrow & \sqrt{-1}\mathfrak{u}(d) \\ & \varphi & \longmapsto & \left(\sum_{h\alpha = i} \varphi_\alpha \varphi_\alpha^* - \sum_{t\alpha = i} \varphi_\alpha^* \varphi_\alpha\right)_{i \in V}. \end{array}$$

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- By results of King [Kin94] and Hoskins [Hos14], there is a homeomorphism
$$\mathcal{M}^{\theta-ss}(Q,d) \simeq \mu^{-1}(\theta)/\mathrm{U}(d).$$

Hitchin-Kobayashi correspondence for quiver bundles

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Theorem ([ACGP03])

A quiver bundle $\mathcal{E} = ((\mathcal{E}_i)_{i \in V}, (\varphi_\alpha)_{\alpha \in E})$ is θ-polystable if there exist an hermitian metric H_i on each $\mathcal{E}_i$ such that

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- *F_i:=Curvature of the Chern connection associated to H_i.*
- *$\Lambda : \Omega^{(i,j)}(X) \to \Omega^{(i-1,j-1)}(X)$ contraction operator w.r.t. a fixed Kähler form on X.*

Applications

Polystability of tensor products of quiver bundles

Theorem

Let $\mathcal{E}', \mathcal{E}''$ be θ' and θ''-polystable quiver bundles respectively. Then, $\mathcal{E}' \otimes \mathcal{E}''$ is θ-polystable for $\theta = (\theta'_i + \theta''_j)_{(i,j) \in V' \times V''}$.

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Corolllary

Tensor products of polystable quiver representations is polystable.

Segre embedding for quiver representations

- Does every polystable representation of $Q' \otimes Q''$ come from the tensor product of polystable representations of Q' and Q''?

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- What sort of analytic/algebraic structure does this set of "decomposable tensors" have?
- For vector spaces we know the answer: the Segre embedding.

Segre embedding for quiver representations (II)

- Choose stability parameters θ', θ'' and θ such that the corresponding symplectic reductions are complex Kähler manifolds.

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- Our previous result shows that there is a well-defined equivariant map:

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- So we have a map

$$F : \mathcal{M}^{\theta'-s}(Q', d') \times \mathcal{M}^{\theta''-s}(Q'', d'') \longrightarrow \mathcal{M}^{\theta-s}(Q, d)$$

Segre embedding for quiver representations (III)

Theorem

The map F is an embedding with image a submanifold of real dimension

$$-2(\langle d', d' \rangle_{Q'} + \langle d'', d'' \rangle_{Q''}).$$

Recall that

$$\langle d, d' \rangle_Q = \sum_{i \in V} d_i d'_i - \sum_{\alpha \in E} d_{t\alpha} d_{h\alpha}$$

is the Euler form of the quiver Q.

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Remark

Under suitable hypothesis (e.g genericity of parameter and Q', Q'' being quivers without cycles) the map F is algebraic and, in fact, a closed immersion.

Recovering classical Segre embedding from the quiver one

- $\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "z_1"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right, "z_n"'] & \mathbb{C} \end{tikzcd} \overset{\theta'=(-1,1)}{\Longrightarrow} \mathcal{M}^{\theta'-s}(Q_n, d') \simeq \mathbb{P}^{n-1}_{\mathbb{C}}$.

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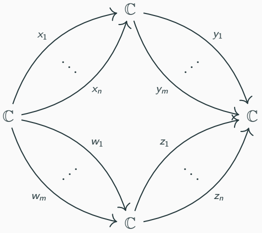
- 

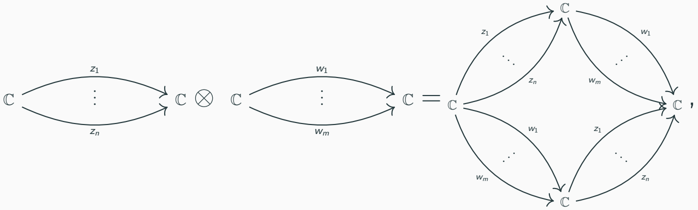
$\overset{\theta=(-2,0,0,2)}{\Longrightarrow} \mathcal{M}^{\theta-s}(Q_n \otimes Q_m, d)$ is the smooth projective subvariety of $\mathbb{P}^{2nm-1}_{\mathbb{C}} \simeq \mathrm{Proj}(\mathbb{C}[s_{ij} = \text{“}x_i y_j\text{”}, t_{kl} = \text{“}w_k z_l\text{”}]_{\substack{i,l=1,\dots,n \\ j,k=1,\dots,m}})$ cutted by the equations

$$\begin{cases} s_{i_1 j_1} s_{i_2 j_2} - s_{i_1 j_2} s_{i_2 j_1} = 0, \\ t_{k_1 l_1} t_{k_2 l_2} - t_{k_1 l_2} t_{k_2 l_1} = 0. \end{cases}$$

Recovering classical Segre embedding from the quiver one (II)

By our previous results:

- The map $\mathcal{M}^{\theta'-s}(Q_n, d') \times \mathcal{M}^{\theta'-s}(Q_m, d') \to \mathcal{M}^{\theta-s}(Q_n \otimes Q_m, d)$ induced by tensorization of representations:



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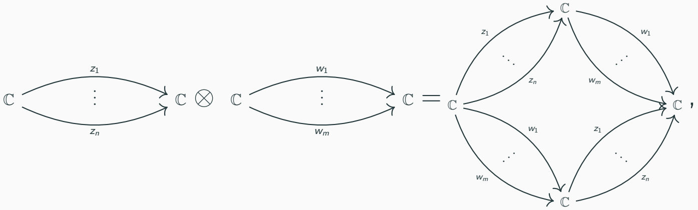
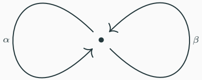


  is a closed immersion.
- Tensorization gives rise to representations with relations. These are encoded by the equations

$$s_{ij} - t_{ji} = 0, i = 1, \ldots, n, j = 1, \ldots, m.$$

  The subvariety of $\mathcal{M}^{\theta-s}(Q_n \otimes Q_m, d)$ cutted by these is that which the classical Segre embedding describes.

Tensors and character varieties



The tensor quiver $Q := Q_J \otimes Q_J$

- Let $\mathcal{R} = \{\alpha\beta - \beta\alpha\}$ and

$$\mathcal{M}(Q, d, \mathcal{R}) = \mathrm{Rep}(Q, d, \mathcal{R}) /\!\!/ \mathrm{GL}(d) \overset{\textit{closed}}{\hookrightarrow} \mathrm{Rep}(Q, d) /\!\!/ \mathrm{GL}(d).$$

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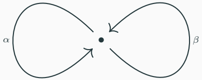

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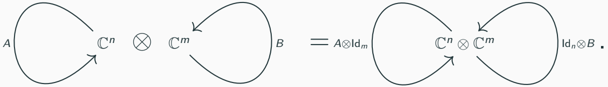
- For $\mathcal{M}(Q_J, n) = \mathrm{Rep}(Q_J, n) /\!\!/ \mathrm{GL}(n)$, there is a map

$$\mathcal{M}(Q_J, n) \times \mathcal{M}(Q_J, m) \longrightarrow \mathcal{M}(Q, d, \mathcal{R})$$

  of affine schemes induced by tensorization of representations:



Tensors and character varieties (II)

- The $\mathrm{GL}(n)$ character variety of $\mathbb{Z}^r$ is the GIT quotient:

$$\begin{aligned}\mathcal{M}_{r,n} &:= \mathrm{Hom}(\mathbb{Z}^r, \mathrm{GL}(n)) /\!\!/ \mathrm{GL}(n) \\ &= \{(A_1, \ldots, A_r) \in \mathrm{GL}(d)^r \mid [A_k, A_j] = 0\} /\!\!/ \mathrm{GL}(d).\end{aligned}$$

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- $\mathcal{M}_{1,n} \overset{\substack{\textit{Distinguished}\\ \textit{open}}}{\hookrightarrow} \mathcal{M}(Q_J, n)$ and $\mathcal{M}_{2,mn} \overset{\substack{\textit{Distinguished}\\ \textit{open}}}{\hookrightarrow} \mathcal{M}(Q, mn, \mathcal{R})$

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  of affine schemes.
- $\mathcal{M}_{1,n} \times \mathcal{M}_{1,m}$ is irreducible so the set-theoretic image of this morphism is irreducible.

Tensors and character varieties (III)

- The scheme theoretic image is the affine closed subscheme of the character variety $\mathcal{M}_{2,mn}$ determined by the kernel of the ring map

$$\mathbb{C}[\mathcal{M}_{2,mn}] \to \mathbb{C}[\mathcal{M}_{1,n}] \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{M}_{1,m}]$$

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  associated to the morphism above.
- As a topological space, the scheme-theoretic image is the closure of the set-theoretic one.
- A similar strategy can be used to obtain distinguished closed subschemes of character varieties from morphisms

$$\mathcal{M}_{1,n_1} \times \cdots \times \mathcal{M}_{1,n_k} \to \mathcal{M}_{k,n_1 \ldots n_k}.$$

See more in: arXiv:2503.11606 !

Questions?

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