The Geometry of Gerbes over Spheres

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A la mujer más sabia, luchadora y maravillosa que he conocido. A ti Merceditas, que siempre has creído en mi y que con tu amor incondicional hiciste posible el alcance de esta meta.

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Introduction

Gerbes where originally introduced by Jean Giraud as a tool for studying non commutative cohomology in Algebraic Geometry. Recently, this objects have developed an important role in Differential Geometry leading to applications on mathematical physics. Basically, a gerbe can be realized as a class in the Čech cohomology group, $\check{H}^3(M, \mathbb{Z})$, so the main goal of this work is to give a meaning to this identification and to give examples of gerbes over spheres.

Chapter 1 is commited to understand the other creatures in the hierarchy to which gerbes belong. That is, understanding what are the geometric objects represented by the Čech cohomology groups in degrees one and two. Of course, we will start by reviewing the main aspects of Čech cohomology and then we will show that the objects represented by the first two cohomology groups are $\mathbb{C}^*$-valued smooth functions and line bundles respectively.

In Chapter 2, we dive deep into the geometry of line bundles. We will find out that it is possible to differentiate the analogous of vector fields (sections) in these objects by defining connections and that there is a notion of curvature that we can associate to line bundles. This will make our way to do differential geometry with gerbes as well.

The last chapter of this work is focused on giving the so promised definition of gerbe, giving examples of this object over spheres and in making differential geometry work on gerbes. To do this, we will use and see the importance of all the tools developed in the previous chapters because locally, as we will see, a gerbe is no more than a line bundle as well as a line bundle can be locally understood as a $\mathbb{C}^*$-valued function.

Chapter 1

Čech Cohomology and the Geometric Objects Associated to Differentiable Manifolds

Given a smooth manifold M and an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of it, Čech cohomology associates to M geometric objects which, in different degrees, can have a completely different appearance. In this chapter we review the main aspects of Čech cohomology and we study the corresponding geometric objects on M in degree one and two (complex-valued functions and line bundles, respectively), the notion of triviality for such objects and the relationship among them.

1.1 Čech Cohomology

Definition 1.1.1. Let M be a manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ an open covering of M. We say that $\mathcal{U}$ is *contractible* if the simplex $U_{i_0\dots i_k} = U_{i_0} \cap \dots \cap U_{i_k}$ is contractible for every $k \in \mathbb{N}$ and for every $(i_0, \dots, i_k) \in I^{k+1}$ such that $U_{i_0\dots i_k} \neq \varnothing$.

Remark 1.1.1. Recall that an open cover $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if for all $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U$. It is well known, using results from Riemannian Geometry, that every open covering of a manifold has a contractible refinement, so that every manifold has a contractible open cover [dC92].

Definition 1.1.2. Let M be a manifold and $\mathcal{U} = \{U_i\}_{i\in I}$ an open covering of M. A *k-chain* is a function from the set of all k-simplices into $\mathbb{Z}$, i.e, a function given by $(i_0, \dots, i_k) \mapsto a_{i_0\dots i_k} \in \mathbb{Z}$ for every $(i_0, \dots, i_k)$ such that $U_{i_0\dots i_k} \neq \varnothing$.

The set of all k-chains, relative to an open cover $\mathcal{U}$, form a group over pointwise addition of functions. We denote this group by $C^k(\mathcal{U}, \mathbb{Z})$. Now, let $\delta : C^k(\mathcal{U}, \mathbb{Z}) \to C^{k+1}(\mathcal{U}, \mathbb{Z})$ be the function defined, for $a \in C^k(\mathcal{U}, \mathbb{Z})$, by

$$(\delta a)_{i_0,\dots,i_{k+1}} = \sum_{j=0}^{k+1} (-1)^j a_{i_0\dots\hat{i}_j\dots i_{k+1}},$$

and which is clearly a group homomorphism. Furthermore, $\delta^2 = 0$ since for $a \in C^k(\mathcal{U}, \mathbb{Z})$:

$$
\begin{aligned}
&\delta(\delta a)_{i_0,\dots,i_{k+2}} = \sum_{j=0}^{k+2} (-1)^j (\delta a)_{i_0,\dots,\hat{i}_j,\dots,i_{k+2}} \\
&= \sum_{j=0}^{k+2} (-1)^j \left(\sum_{0 \le m < j \le k+2} (-1)^m a_{i_0 \dots \hat{i}_m \dots \hat{i}_j \dots i_{k+2}} + \sum_{0 \le j < m \le k+2} (-1)^{m-1} a_{i_0 \dots \hat{i}_j \dots \hat{i}_m \dots i_{k+2}} \right) \\
&= \sum_{j=0}^{k+2} \sum_{0 \le m < j \le k+2} (-1)^{m+j} a_{i_0 \dots \hat{i}_m \dots \hat{i}_j \dots i_{k+2}} - \sum_{j=0}^{k+2} \sum_{0 \le j < m \le k+2} (-1)^{m+j} a_{i_0 \dots \hat{i}_j \dots \hat{i}_m \dots i_{k+2}} \\
&= 0.
\end{aligned}
$$

This implies, for $k \geq 0$, that the sequence of groups

$$
C^0(\mathcal{U}, \mathbb{Z}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathbb{Z}) \xrightarrow{\delta} \dots \xrightarrow{\delta} C^k(\mathcal{U}, \mathbb{Z}) \xrightarrow{\delta} C^{k+1}(\mathcal{U}, \mathbb{Z}) \xrightarrow{\delta} \dots
$$

satisfies that $\mathrm{im}(\delta_{k-1}) \subseteq \ker(\delta_k)$, where δ_k denotes the map $\delta : C^k(\mathcal{U}, \mathbb{Z}) \to C^{k+1}(\mathcal{U}, \mathbb{Z})$. The *k-th cohomology group* $\ker(\delta_k)/\mathrm{im}(\delta_{k-1})$, relative to an open cover $\mathcal{U}$, is denoted $H^k(\mathcal{U}, \mathbb{Z})$. Observe that if $\mathcal{V}$ is a refinement of $\mathcal{U}$, there is a canonical homomorphism $H^k(\mathcal{V}, \mathbb{Z}) \to H^k(\mathcal{U}, \mathbb{Z})$ and, since the set of open covers of a manifold is a directed set under refinement, we are allowed to take the direct limit.

Definition 1.1.3. The *k-th Čech cohomology group* of a differentiable manifold M is given by:

$$
\check{H}^k(M, \mathbb{Z}) = \varinjlim_{\mathcal{U}} H^k(\mathcal{U}, \mathbb{Z}).
$$

Remark 1.1.2. For any open cover $\mathcal{U}$ there is canonical homomorphism $H^k(\mathcal{U}, \mathbb{Z}) \to \check{H}^k(M, \mathbb{Z})$. Moreover, if $\mathcal{U}$ is contractible, this homomorphism is actually an isomorphism.

We want to point out that instead of using $\mathbb{Z}$ to construct the Čech cohomology groups we could have used any abelian group by considering maps from the set of all k-simplices into G, where G is an arbitrary abelian group.

There is one last thing to add in this section. Let $C^\infty(\mathbb{C})$, $C^\infty(\mathbb{C}^*)$ be the sheaves of $\mathbb{C}$-valued and $\mathbb{C}^*$-valued smoooth functions respectively over a differentiable manifold M. In [Bry08] is shown that the sequence of sheaves

$$
0 \longrightarrow \mathbb{Z} \longrightarrow C^\infty(\mathbb{C}) \xrightarrow{\exp} C^\infty(\mathbb{C}^*) \longrightarrow 1
$$

is exact, meaning that $\check{H}^{k+1}(M, \mathbb{Z}) \cong \check{H}^k(M, C^\infty(\mathbb{C}^*))$ for $k \geq 0$. In what follows we shall see that in some cases is better to think on elements in $\check{H}^k(M, C^\infty(\mathbb{C}^*))$ rather than in $\check{H}^{k+1}(M, \mathbb{Z})$, for instance, when defining a gerbe.

1.2 Čech Cohomology and Complex Functions on Manifolds

This section is committed to understand what geometric object represents an element in the first Čech cohomology group $\check{H}^1(M, \mathbb{Z})$, for a differentiable manifold M.

Let M be a differentiable manifold and $f : M \to \mathbb{C}^*$ a smooth function. By remark 1.1.1, there exists a contractibe open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M. Notice that f can be built up from local functions $g_\alpha := f\,|_{U_\alpha} \colon U_\alpha \to \mathbb{C}^*$, for $\alpha \in I$, that satisfy the following condition

$$g_\alpha g_\beta^{-1} = 1 \tag{1.2.0.1}$$

on $U_{\alpha\beta} = U_\alpha \cap U_\beta$. We will refer to this condition as a *cocycle condition* on double intersections. Defining $f_\alpha : U_\alpha \to \mathbb{C}$ as

$$f_\alpha = \frac{1}{2\pi i} \log g_\alpha, \tag{1.2.0.2}$$

for all $\alpha \in I$, we observe that whenever $U_{\alpha\beta} \neq \varnothing$, taking the differences $a_{\alpha\beta} = f_\alpha - f_\beta$ one has

$$\begin{aligned} \exp(2\pi i a_{\alpha\beta}) &= \exp(\log g_\alpha - \log g_\beta) \\ &= \exp(\log g_\alpha g_\beta^{-1}) \\ &= 1 \end{aligned}$$

using the cocycle condition. This implies that $a_{\alpha\beta}$ must be $\mathbb{Z}$-valued and since continuous, it must be constant on $U_{\alpha\beta}$. Hence, we have defined an element $a \in C^1(\mathcal{U}, \mathbb{Z})$. Moreover, note that

$$\begin{aligned} (\delta a)_{\alpha\beta\gamma} &= a_{\beta\gamma} - a_{\alpha\gamma} + a_{\alpha\beta} \\ &= (f_\beta - f_\gamma) - (f_\alpha - f_\gamma) + (f_\alpha - f_\beta) \\ &= 0. \end{aligned}$$

Thus, $a \in \ker(\delta_1)$ giving rise to an element $[a] \in \check{H}^1(M, \mathbb{Z})$.

At this point we see that is possible to build elements in the group $\check{H}^1(M, \mathbb{Z})$ from *global complex-valued functions* on M, i.e. smooth functions $f : M \to \mathbb{C}^*$. In what is left of this section we will show that in fact, there is a bijective correspondence between these functions and the first Čech cohomology group.

Let $F_M = \{f : M \to \mathbb{C}^* | f \text{ is smooth}\}$ and $\sim$ be the equivalence relation over F_M given by $f \sim g$ if and only if $f = \lambda g$ for some $\lambda \in \mathbb{C}^*$.

Proposition 1.2.1. $\mathcal{F}(M) = F_M/\sim$ is an abelian group having as group operation $([f], [g]) \mapsto [fg]$ and identity element the class of all the constant functions $f : M \to \{c\}$, for $c \in \mathbb{C}^*$.

Remark 1.2.1. The map $a \in C^1(\mathcal{U}, \mathbb{Z})$ defined above only depends on the class $[f] \in \mathcal{F}(M)$. Indeed, if f^1, f^2 are global functions such that $f^1 \sim f^2$ then one has that $f^2 = \lambda f^1$ for $\lambda \in \mathbb{C}^*$. In consequence

$$\begin{aligned}
f^2_\alpha - f^2_\beta &= \frac{1}{2\pi i}\log g^2_\alpha - \frac{1}{2\pi i}\log g^2_\beta \\
&= \frac{1}{2\pi i}\log \lambda g^1_\alpha - \frac{1}{2\pi i}\log \lambda g^1_\beta \\
&= \frac{1}{2\pi i}(\log \lambda + \log g^1_\alpha) - \frac{1}{2\pi i}(\log \lambda + \log g^1_\beta) \\
&= \frac{1}{2\pi i}\log g^1_\alpha - \frac{1}{2\pi i}\log g^1_\beta \\
&= f^1_\alpha - f^1_\beta = a_{\alpha\beta} \in \mathbb{Z}.
\end{aligned}$$

Theorem 1.2.1. Let M be a smooth manifold. Then, $\mathcal{F}(M) \cong \check{H}^1(M, \mathbb{Z})$.

Proof. Let $\kappa : \mathcal{F}(M) \to \check{H}^1(M, \mathbb{Z})$ be the group homomorphism defined as $[f] \mapsto [a]$ where $a_{\alpha\beta} = f_\alpha - f_\beta$ and f_α is as in equation (1.2.0.2). First, we will prove the surjectivity of the map κ. Choose an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M that is both contractible and locally finite and a partition of unity $\sum_{i\in I} h_i = 1$ relative to this cover where $\mathrm{supp}(h_i) \subseteq U_i$ for all $i \in I$. Let $a \in C^1(\mathcal{U}, \mathbb{Z})$ such that $a \in \ker(\delta_1)$. Define $f_\alpha : U_\alpha \to \mathbb{C}$ as

$$f_\alpha = \sum_{\gamma \in I} a_{\alpha\gamma} h_\gamma$$

and observe that

$$f_\alpha - f_\beta = \sum_{\gamma\in I} a_{\alpha\gamma}h_\gamma - \sum_{\gamma\in I} a_{\beta\gamma}h_\gamma = \sum_{\gamma\in I}(a_{\alpha\gamma} - a_{\beta\gamma})h_\gamma = \sum_{\gamma\in I} a_{\alpha\beta}h_\gamma = a_{\alpha\beta} \in \mathbb{Z}.$$

Now, put $g_\alpha = \exp(2\pi i f_\alpha)$ and observe that the equality (1.2.0.1) holds. It follows trivially that there exists a global function $f : M \to \mathbb{C}^*$ such that $f|_{U_\alpha} = g_\alpha$ for all $\alpha \in I$.

Finally, we will show that κ is injective. Let f^1, f^2 be two global functions such that $\kappa([f^1]) = \kappa([f^2])$ and $\mathcal{U}$ the open cover given above. Note that we can choose logarithms such that $f_\alpha - f_\beta = 0$ for $f_\alpha = f^1_\alpha - f^2_\alpha$ and

$$f^k_\alpha = \frac{1}{2\pi i}\log g^k_\alpha$$

where $k = 1, 2$. This relation yields

$$\frac{g^1_\alpha}{g^2_\alpha} = \frac{g^1_\beta}{g^2_\beta}$$

for all $\alpha, \beta \in I$ which implies that $f_1 \sim f_2$ as desired.

□

In summary, $\check{H}^1(M,\mathbb{Z})$ represents the set of all global functions up to the equivalence relation $\sim$. Given this construction one may think the *trivial* object in this category as any constant global function, i.e. a function of the form $f : M \to \{c\}$ for $c \in \mathbb{C}^*$, and the *non trivial* object as any non constant global function. Later on we shall see the importance of this distinction.

Example 1.2.1. (*Global Functions over* $\mathbb{S}^1$*,* $\mathbb{S}^2$ *and* $\mathbb{S}^3$) It is well known, using results from Algebraic Topology, that $\check{H}^k(\mathbb{S}^n,\mathbb{Z})$ is trivial for all $k \neq n$. As a matter of fact, $\check{H}^n(\mathbb{S}^n,\mathbb{Z}) \cong \mathbb{Z}$. Hence, the only possible existing global functions over $\mathbb{S}^2$ and $\mathbb{S}^3$ are the trivial ones,i.e. constant functions.

The fact of $\check{H}^1(\mathbb{S}^1,\mathbb{Z})$ being isomorphic to $\mathbb{Z}$ implies that there will be global function over $\mathbb{S}^1$ for each integer. What is more, since $\mathbb{Z}$ is cyclic, there is an unique global function over $\mathbb{S}^1$ that generates all the others.

1.3 Čech Cohomology and Line Bundles on Manifolds

We move one step forward into the hierarchy. In this section we study the relationship between line bundles and $\check{H}^2(M,\mathbb{Z})$ for a given differentiable manifold M. But, unlike section 1.2, we dive deep into the structure of these objects and its geometry.

1.3.1 Line Bundles

In first place, we will develop the classic theory of vector bundles, focused on the objects we are interested in, which are line bundles over a differentiable manifold M. This is way easier than tackling the definition of a line bundle from the cohomology perspective, i.e as an equivalence class in $\check{H}^2(M,\mathbb{Z})$.

Definition 1.3.1.1. Let M be a differentiable manifold. A *line bundle* L over M is a vector bundle whose fibers are isomorphic to $\mathbb{C}$, i.e. a differentiable manifold together with a C^∞ map $\pi : L \to M$ such that:

- For all $p \in M$, $L_p = \pi^{-1}(p)$ has structure of one dimensional complex vector space.
- (*Local Trivialization*) For all $p \in M$, there exists an open set U, containing p, and a diffeomorphism φ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times \mathbb{C} \arrow[dl, "\mathrm{pr}_U"] \\ U & \end{tikzcd}$$

Moreover, the map $\varphi|_{L_p} : L_p \to \{p\} \times \mathbb{C}$ is a linear isomorphism.

Definition 1.3.1.2. Two line bundles, L, J over a manifold M are said to be *isomorphic* if there is a diffeomorphism $\varphi : L \to J$ such that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[d, "\pi_L"] & J \arrow[dl, "\pi_J"] \\ M & \end{tikzcd}$$

Moreover, $\varphi|_{L_p} : L_p \to J_p$ is a linear isomorphism between one dimensional vector spaces over $\mathbb{C}$. We write $L \cong J$ when L and J are isomorphic.

Remark 1.3.1.1. A line bundle L over a differentiable manifold M is said to be *trivial* if $L \cong M \times \mathbb{C}$. That is why we talk about "local trivializations" on definition 1.3.1.1.

Definition 1.3.1.3. Let M be a differentiable manifold and L a line bundle over M. A section of L is a C^∞ function $\sigma : M \to L$ such that $\pi \circ \sigma = \mathrm{id}_M$. The set of all sections of a line bundle is clearly a vector space over $\mathbb{C}$ and is denoted by $\Gamma(M, L)$.

Proposition 1.3.1.1. A line bundle L over a differentiable manifold M is trivial if and only if it admits a nowhere vanishing section.

Proof. Let L be a line bundle over a differentiable manifold M.
"$\Rightarrow$" Suppose that L is trivial. By definition 1.3.1.2, there exists a diffeomorphism $\varphi : L \to M \times \mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[dr, "\pi_L"] & M \times \mathbb{C} \arrow[d, "\mathrm{pr}_M"] \\ & M \end{tikzcd}$$

Let $\sigma' : M \to M \times \mathbb{C}$ be the section defined by $p \mapsto (p, 1)$. Note that under the conmmutativity of the diagram, the function $\sigma := \varphi^{-1} \circ \sigma' : M \to L$ satisfies

$$\pi_L \circ \sigma = \mathrm{pr}_M \circ \varphi \circ \varphi^{-1} \circ \sigma' = \mathrm{pr}_M \circ \sigma' = \mathrm{id}_M;$$

hence, it is a section. Moreover, $\sigma(p) \neq 0_{L_p}$ for all $p \in M$ since $\sigma(p) = \varphi^{-1} \circ \sigma'(p) = \varphi^{-1}(1, p) \neq 0_{L_p}$ due to the fact that φ^{-1} is a linear isomorphism fiberwise, stated in definition 1.3.1.2.

"$\Leftarrow$" Now, suppose that L has a nowhere vanishing section σ, i.e. for all $p \in M$, $\sigma(p) \neq 0_{L_p}$. Let us define $\varphi : M \times \mathbb{C} \to L$ as $(p, \lambda) \mapsto \lambda\sigma(p)$ which is clearly a C^∞ function.

First, we will show that φ is linear fibwerwise. Fixing $p \in M$, $\alpha \in \mathbb{C}$ and $(p, \lambda_1), (p, \lambda_2) \in (M \times \mathbb{C})_p$ we notice that

$$\begin{aligned}\varphi(\alpha(p, \lambda_1) + (p, \lambda_2)) &= \varphi((p, \alpha\lambda_1) + (p, \lambda_2)) \\ &= \varphi(p, \alpha\lambda_1 + \lambda_2) = (\alpha\lambda_1 + \lambda_2)\sigma(p) \\ &= \alpha\lambda_1\sigma(p) + \lambda_2\sigma(p) = \alpha\varphi(p, \lambda_1) + \varphi(p, \lambda_2)\end{aligned}$$

showing that φ is linear fiberwise.

Secondly, we will check that φ is bijective. Let $(p, \lambda) \in \ker(\varphi|_{(M\times\mathbb{C})_p})$, then, $\varphi(p, \lambda) = \lambda\sigma(p) = 0_{L_p}$. But, $\sigma(p) \neq 0_{L_p}$ since σ is a nowhere vanishing concluding that $\lambda = 0$ and that in consequence, $\varphi|_{(M\times\mathbb{C})_p}$ is injective. On the other hand, let $q \in L$, then, there exists $p \in M$ such that $q \in L_p$. A basis for L_p is clearly given by the set $\{\sigma(p)\}$ which implies that $q = \lambda\sigma(p)$ for some $\lambda \in \mathbb{C}$. Hence, $\varphi(p, \lambda) = q$ concluding that φ is surjective.

Until now we have proven that φ is a vector space isomoprhism fiberwise. Thus, φ is a bijection between L and $M \times \mathbb{C}$ with inverse φ^{-1} given by $\lambda\sigma(p) \mapsto (\pi(\lambda\sigma(p)), \lambda) = (p, \lambda)$ that is C^∞. Therefore, φ is a diffeomorphism between L and $M \times \mathbb{C}$ concluding that L is trivial. $\square$

Remark 1.3.1.2. By proposition 1.3.1.1, the set of local trivializations $\{\varphi_\alpha : U_\alpha \times \mathbb{C} \to \pi^{-1}(U_\alpha)\}_{\alpha\in I}$ of a line bundle induce a set $\{\sigma_\alpha : U_\alpha \to L\}_{\alpha\in I}$ of nowhere vanishing sections and viceversa. So, one may think a line bundle as a set $\{(U_\alpha, \sigma_\alpha)\}_{\alpha\in I}$, where $\{U_\alpha\}_{\alpha\in I}$ is an open cover of M and $\sigma_\alpha : U_\alpha \to L$ is a nowhere vanishing section for all $\alpha \in I$, that "glues" together in a nice way on the non empty double intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$. From now on, our work is going to be concentrated on formalizing these ideas.

Given a line bundle L over a differentiable manifold M and two local trivializations, $\varphi_\alpha : U_\alpha \times \mathbb{C} \to \pi^{-1}(U_\alpha), \varphi_\beta : U_\beta \times \mathbb{C} \to \pi^{-1}(U_\beta)$ we will have the diffeomorphism $\varphi_\beta \circ \varphi_\alpha^{-1} : \pi^{-1}(U_{\alpha\beta}) \to \pi^{-1}(U_{\alpha\beta})$ whenever $U_{\alpha\beta} \neq \varnothing$. Observe that, by definition 1.3.1.2, $\varphi_\beta \circ \varphi_\alpha^{-1}$ is linear isomorphism fiberwise, hence the following diagram must commute for all $p \in U_{\alpha\beta}$:

$$\begin{tikzcd} L_p \arrow[r, "\varphi_\beta\circ\varphi_\alpha^{-1}"] \arrow[d] & L_p \\ \mathbb{C} \arrow[r, "g_{\beta\alpha}(p)"] & \mathbb{C} \arrow[u] \end{tikzcd}$$

where $g_{\beta\alpha}(p)$ is the basis change matrix (transformation matrix) with respect to the basis $\sigma_\alpha(p)$ and $\sigma_\beta(p)$ of L_p. Moreover, $g_{\beta\alpha}(p) \in \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$ and

$$\sigma_\alpha(p) = g_{\beta\alpha}(p)\sigma_\beta(p) \tag{1.3.1.1}$$

which implies that $g_{\beta\alpha}$ depends smoothly on p.

Definition 1.3.1.4. The smooth maps $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$ are called *transition functions* of L and for convenience, in some cases we will denote $g_{\alpha\beta}(p)$ as $g_{\alpha\beta}$.

Proposition 1.3.1.2. The transition functions of a line bundle L satisfy the following properties:

1. $g_{\alpha\alpha} = 1$.

2. $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$.

3. (*Cocycle Condition*) $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ over $U_{\alpha\beta\gamma}$ or equivalently $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$.

Proof. 1 and 2 are trivial. Now, consider the map $\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ \varphi_\gamma^{-1} \circ \varphi_\gamma \circ \varphi_\alpha^{-1} : \pi^{-1}(U_{\alpha\beta\gamma}) \to \pi^{-1}(U_{\alpha\beta\gamma})$ which is clearly defined by $\lambda\sigma_\alpha(p) \mapsto \lambda g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}\sigma_\alpha(p)$. But, also we have that $\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ \varphi_\gamma^{-1} \circ \varphi_\gamma \circ \varphi_\alpha^{-1} = \mathrm{id}|_{\pi^{-1}(U_{\alpha\beta\gamma})}$ which implies that $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$ as desired. $\square$

In summary, we can understand a line bundle in multiple ways. The one as in definition 1.3.1.1. Also, as a set of nowhere vanishing sections on an open cover of M which glue together through the so called transition functions defined in definition 1.3.1.4. Furthermore, in the next proposition we will see that it is enough to have an open cover of M together with a set of transition functions to define a line bundle.

Proposition 1.3.1.3. Let M be a smooth manifold, $\{U_\alpha\}_{\alpha\in I}$ an open cover of M and $\{g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*\}$ a set of transition functions such that the equalities in proposition 1.3.1.2 hold. Then, $L = \bigsqcup_{\alpha\in I} U_\alpha \times \mathbb{C}/\sim$ is a line bundle over M where $\sim$ is the equivalence relation defined by $(p, \lambda_1) \sim (q, \lambda_2)$ if and only if $p = q$ and $\lambda_2 = g_{\beta\alpha}\lambda_1$.

This is not a surprise at all, recall that a line bundle is locally trivial so we only care about gluing properly on double intersections these trivial pieces. This is, in fact, the idea we are going to capture when relating line bundles to Čech cohomology; that a line bundle is uniquely determined by its transition functions.

1.3.2 The Picard Group and the Second Čech Cohomology Group

In definition 1.3.1.2 we discussed under what circumstances two line bundles over a differentiable manifold M where isomorphic. Clearly, the fact of being isomorphic defines a equivalence relation on $\mathcal{L}_M$, the set of all line bundles over a differentiable manifold M.

Proposition 1.3.2.1. Let $\mathrm{Pic}(M) = \mathcal{L}_M/\cong$. Then, $\mathrm{Pic}(M)$ has abelian group structure by taking tensor products of line bundles. $\mathrm{Pic}(M)$ is called *the picard group of M*.

Proof. We define $\otimes : \mathrm{Pic}(M)\times\mathrm{Pic}(M) \to \mathrm{Pic}(M)$ as $([L^1],[L^2]) \to [L^1\otimes L^2]$. Here, $L^1 \otimes L^2 = \bigsqcup_{p\in M} L^1_p \otimes_{\mathbb{C}} L^2_p$ which is well defined under the equivalence relation $\cong$. The identity element of $\mathrm{Pic}(M)$ corresponds to the trivial bundle equivalence class, $[M \times \mathbb{C}]$, since for all $L \in \mathcal{L}_M$,

$$(M \times \mathbb{C}) \otimes L = \bigsqcup_{p\in M} (M \times \mathbb{C})_p \otimes_{\mathbb{C}} L_p = \bigsqcup_{p\in M} \mathbb{C} \otimes_{\mathbb{C}} L_p = \bigsqcup_{p\in M} L_p = L.$$

The binary operation, $\otimes$, defined above is clearly associative since the tensor product of complex vector spaces is associative. In order to check the existence of inverse elements on $\mathrm{Pic}(M)$ we recall the dual bundle of L, which is given by $L^* = \bigsqcup_{p\in M} L^*_p$. Note that

$$L \otimes L^* = \bigsqcup_{p\in M} L_p \otimes_{\mathbb{C}} L^*_p \cong \bigsqcup_{p\in M} \mathrm{End}(L_p) = \bigsqcup_{p\in M} \mathbb{C} = M \times \mathbb{C}.$$

Hence, $[L]^{-1} = [L^*]$. It only remains to verify the commutativity of $\mathrm{Pic}(M)$ but this follows immediately from the fact that the tensor product of complex vector spaces is commutative. □

Our labor, in what remains of this section, will be to construct an isomorphism between the Picard group and the second Čech Cohomology group.
Let M be a smooth manifold and L a line bundle over M. By remarks 1.1.1 and 1.3.1.2 we may choose a contractible cover of M together with a set of nowhere vanishing sections, i.e, a set $\{(U_\alpha, \sigma_\alpha)\}_{\alpha\in I}$ that defines the line bundle L. Now, define for all $\alpha, \beta \in I$ such that $U_{\alpha\beta} \neq \varnothing$, $f_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}$ as

$$f_{\alpha\beta} = \frac{1}{2\pi i} \log g_{\alpha\beta} \tag{1.3.2.1}$$

where $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$ are the corresponding transition functions associated to L. Whenever $U_{\alpha\beta\gamma} \neq \varnothing$, observe that for $a_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma}$ one has

$$\begin{aligned}\exp(2\pi i\ a_{\alpha\beta\gamma}) &= \exp(2\pi i(f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma}))\\ &= \exp(\log g_{\alpha\beta} + \log g_{\beta\gamma} - \log g_{\alpha\gamma})\\ &= \exp(\log(g_{\alpha\beta} g_{\beta\gamma} g_{\alpha\gamma}^{-1})) = \exp(\log(g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha}))\\ &= 1\end{aligned}$$

using the cocycle condition stated in proposition 1.3.1.2. This implies that $a_{\alpha\beta\gamma}$ must be $\mathbb{Z}-$valued, but since continuous, it must be constant on $U_{\alpha\beta\gamma}$. Therefore, a is an element in $C^2(\mathcal{U}, \mathbb{Z})$. Also, note that

$$\begin{aligned}&(\delta a)_{\alpha\beta\gamma\delta} = a_{\beta\gamma\delta} - a_{\alpha\gamma\delta} + a_{\alpha\beta\delta} - a_{\alpha\beta\gamma}\\ &= (f_{\beta\gamma} + f_{\gamma\delta} - f_{\beta\delta}) - (f_{\alpha\gamma} + f_{\gamma\delta} - f_{\alpha\delta}) + (f_{\alpha\beta} + f_{\beta\delta} - f_{\alpha\delta}) - (f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma})\\ &= 0.\end{aligned}$$

Hence, $a \in \ker(\delta_2)$ defining an element $[a] \in \check{H}^2(M, \mathbb{Z})$.

One may think that the map $a \in C^2(\mathcal{U}, \mathbb{Z})$ depends on the choice of $\{(U_\alpha, \sigma_\alpha)\}_{\alpha \in I}$. However, it only depends on $[L] \in \text{Pic}(M)$ by the following proposition.

Proposition 1.3.2.2. Let $g^1_{\alpha\beta}, g^2_{\alpha\beta}$ be the transition functions for the line bundles L^1, L^2 relative to the sets $\{(U_\alpha, \sigma^1_\alpha)\}_{\alpha \in I}, \{(U_\alpha, \sigma^2_\alpha)\}_{\alpha \in I}$. Then, $L^1 \cong L^2$ if and only if there exist smooth functions $\lambda_\alpha : U_\alpha \to \mathbb{C}^*$ such that $\lambda_\alpha g^1_{\alpha\beta} \lambda^{-1}_\beta = g^2_{\alpha\beta}$ on $U_{\alpha\beta}$.

Proof. Let L^1, L^2 be line bundles over a differentiable manifold M and $g^1_{\alpha\beta}, g^2_{\alpha\beta}$ be the corresponding transition functions relative to the sets $\{(U_\alpha, \sigma^1_\alpha)\}_{\alpha \in I}$ and $\{(U_\alpha, \sigma^2_\alpha)\}_{\alpha \in I}$.
"$\Rightarrow$" Suppose that $L^1 \cong L^2$. Then, by definition 1.3.1.2, there is a diffeomorphism, $\varphi : L^1 \to L^2$ which is linear fiberwise. Fix $\alpha, \beta \in I$ such that $U_{\alpha\beta} \neq \varnothing$ and let $p \in U_{\alpha\beta}$. Recall that $\sigma^1_\beta(p)$ is a basis for the one dimensional vector space L^1_p and that φ is a linear isomorphism fiberwise. Hence, φ sends basis into basis and, using equation (1.3.1.1),

$$\varphi(\sigma^1_\beta(p)) = \lambda_\beta(p)\sigma^2_\beta(p) = \lambda_\beta(p) g^2_{\alpha\beta} \sigma^2_\alpha(p)$$

where $\lambda_\beta(p) \in \mathbb{C}^*$. On the other hand, again by equation (1.3.1.1),

$$\varphi(\sigma^1_\beta(p)) = \varphi(g^1_{\alpha\beta}\sigma^1_\alpha(p)) = g^1_{\alpha\beta}\varphi(\sigma^1_\alpha(p)) = g^1_{\alpha\beta}\lambda_\alpha(p)\sigma^2_\alpha(p)$$

and here $\lambda_\alpha(p) \in \mathbb{C}^*$ too. Combining both equalities we get that $\lambda_\alpha(p) g^1_{\alpha\beta} = \lambda_\beta(p) g^2_{\alpha\beta}$ and therefore,

$$\lambda_\alpha(p) g^1_{\alpha\beta} \lambda^{-1}_\beta(p) = g^2_{\alpha\beta}$$

on $U_{\alpha\beta}$. Observe that we may define λ_α all over U_α since the nowhere vanishing section, σ^1_α, gives us a basis, for a one dimensional complex vector space, for each point $p \in U_\alpha$. Moreover, note that $\lambda_\alpha : U_\alpha \to \mathbb{C}^*$ is no more than the restriction of φ to $\pi^{-1}(U_\alpha)$ in the sense that for $x \in \pi^{-1}(U_\alpha)$, we know that there exist $p \in U_\alpha$ and $\lambda \in \mathbb{C}$ such that $x = \lambda\sigma^1_\alpha(p)$ and therefore $\varphi(x) = \varphi(\lambda\sigma^1_\alpha(p)) = \lambda\varphi(\sigma^1_\alpha(p)) = \lambda_\alpha(p)\lambda\sigma^2_\alpha(p)$. From this statement follows the differentiability of λ_α.

"$\Leftarrow$" Now, we assume that there exist smooth functions $\lambda_\alpha : U_\alpha \to \mathbb{C}^*$ such that $\lambda_\alpha g^1_{\alpha\beta} \lambda^{-1}_\beta = g^2_{\alpha\beta}$ on $U_{\alpha\beta}$. Define the smooth function $\varphi : L^1 \to L^2$ as $\lambda\sigma^1_\alpha(p) \mapsto \lambda\lambda_\alpha(p)\sigma^2_\alpha(p)$ which is clearly a linear isomorphism fiberwise. Note that we may have an issue on all the points, $x = \lambda\sigma^1_\alpha(p) = \lambda'\sigma^1_\beta(p) \in \pi^{-1}(U_{\alpha\beta})$, since we do not know if $\varphi(\lambda\sigma^1_\alpha(p)) = \varphi(\lambda'\sigma^1_\beta(p))$. In subsection 1.3.1 we discussed that the two representations of x in the basis $\sigma^1_\alpha(p)$ and $\sigma^1_\beta(p)$ are related via the transition functions by

$$x = \lambda'\sigma^1_\beta(p) = \lambda' g^1_{\alpha\beta}\sigma^1_\alpha(p)$$

where clearly $\lambda = \lambda' g^1_{\alpha\beta}$. Thus, we need to show that $\varphi(\lambda'\sigma^1_\beta(p)) = \varphi(\lambda' g^1_{\alpha\beta}\sigma^1_\alpha(p))$.

Using the hypothesis and equation (1.3.1.1) we get

$$\begin{aligned}\varphi(\lambda'\sigma^1_\beta(p)) &= \lambda'\lambda_\beta(p)\sigma^2_\beta(p)\\ &= \lambda'\lambda_\beta(p)g^2_{\alpha\beta}\sigma^2_\alpha(p)\\ &= \lambda'\lambda_\beta(p)\lambda_\alpha(p)g^1_{\alpha\beta}\lambda^{-1}_\beta(p)\sigma^2_\alpha(p)\\ &= \lambda'\lambda_\alpha(p)g^1_{\alpha\beta}\sigma^2_\alpha(p)\\ &= \varphi(\lambda' g^1_{\alpha\beta}\sigma^1_\alpha(p)) = \varphi(\lambda\sigma^1_\alpha(p))\end{aligned}$$

as desired. In summary, $\varphi : L^1 \to L^2$ is a well defined smooth function, a linear isomorphism fiberwise and a funcion with smooth inverse given by $\lambda\sigma^2_\alpha(p) \mapsto \lambda/\lambda_\alpha(p)\sigma^1_\alpha(p)$. This means that $L^1 \cong L^2$ as we wanted to prove. □

It is now easy to check that a does not depend on the choice of the nowhere vanishing sections nor the line bundle itself, just on the equivalence class $[L] \in \mathrm{Pic}(M)$. For instance, if we choose two isomorphic line bundles L^1, L^2 one has that

$$\begin{aligned}f^2_{\alpha\beta} + f^2_{\beta\gamma} - f^2_{\alpha\gamma} &= \frac{1}{2\pi i}(\log g^2_{\alpha\beta} + \log g^2_{\beta\gamma} - \log g^2_{\alpha\gamma})\\ &= \frac{1}{2\pi i}(\log \lambda_\alpha g^1_{\alpha\beta}\lambda^{-1}_\beta + \log \lambda_\beta g^1_{\beta\gamma}\lambda^{-1}_\gamma - \log \lambda_\alpha g^1_{\alpha\gamma}\lambda^{-1}_\gamma)\\ &= \frac{1}{2\pi i}(\log g^1_{\alpha\beta} + \log g^1_{\beta\gamma} - \log g^1_{\alpha\gamma})\\ &= f^1_{\alpha\beta} + f^1_{\beta\gamma} - f^1_{\alpha\gamma} = a_{\alpha\beta\gamma}.\end{aligned}$$

We conclude presenting the so mentioned main result of this section, were we combine all our work until now.

Theorem 1.3.2.1. Let M be a smooth manifold. Then, $\mathrm{Pic}(M) \cong \check{H}^2(M,\mathbb{Z})$.

Proof. Let $\kappa : \mathrm{Pic}(M) \to \check{H}^2(M,\mathbb{Z})$ be the group homomorphism defined as $[L] \mapsto [a]$ where $a_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma}$ and $f_{\alpha\gamma}$ is as in equation (1.3.2.1).

First, we will prove the surjectivity of the map κ. Choose an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of M to be both contractible and locally finite and a partition of unity, $\sum_{p\in I} h_p = 1$, relative to this cover where $\mathrm{supp}(h_p) \subseteq U_p$. Let $a \in C^2(\mathcal{U},\mathbb{Z})$ such that $a \in \ker(\delta_2)$. Define $f_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}$ as

$$f_{\alpha\beta} = \sum_{p\in I} a_{\alpha\beta p}h_p$$

and observe that

$$f_{\alpha\beta} - f_{\alpha\gamma} + f_{\beta\gamma} = \sum_{p\in I} a_{\alpha\beta p}h_p - \sum_{p\in I} a_{\alpha\gamma p}h_p + \sum_{p\in I} a_{\beta\gamma p}h_p = \sum_{p\in I}(a_{\alpha\beta p} - a_{\alpha\gamma p} + a_{\beta\gamma p})h_p.$$

But, since $a \in \ker(\delta_2)$, $a_{\beta\gamma p} - a_{\alpha\gamma p} + a_{\alpha\beta p} - a_{\alpha\beta\gamma} = 0$. Thus,

$$f_{\alpha\beta} - f_{\alpha\gamma} + f_{\beta\gamma} = \sum_{p\in I}(a_{\alpha\beta p} - a_{\alpha\gamma p} + a_{\beta\gamma p})h_p = \sum_{p\in I} a_{\alpha\beta\gamma}h_p = a_{\alpha\beta\gamma} \in \mathbb{Z}.$$

Now, put $g_{\alpha\beta} = \exp(2\pi i f_{\alpha\beta})$ and note that the equalities in proposition 1.3.1.2 hold. Therefore, by proposition 1.3.1.3, there exists a line bundle L having the $g_{\alpha\beta}$ as transition functions. One then has that $\kappa([L]) = [a]$ proving the surjectivity.

Finally, we will show that κ is injective. Let L^1, L^2 be any two line bundles over M such that $\kappa([L^1]) = \kappa([L^2])$. Again choose a contractible locally finite open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of M. Let $g^1_{\alpha\beta}, g^2_{\alpha\beta}$ be the transition functions for the line bundles L^1, L^2 respectively relative to the sets of nowhere vanishing sections $\{(U_\alpha, \sigma^1_\alpha)\}_{\alpha\in I}, \{(U_\alpha, \sigma^2_\alpha)\}_{\alpha\in I}$. Now, note that we may choose logarithms so that $f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma} = 0$ for $f_{\alpha\beta} = f^1_{\alpha\beta} - f^2_{\alpha\beta}$ and

$$f^k_{\alpha\beta} = \frac{1}{2\pi i}\log g^k_{\alpha\beta}$$

where $k = 1, 2$. Choose again a partition of unity, $\sum_{p\in I} h_p = 1$, relative to the cover $\mathcal{U}$ and define $\psi_\alpha : U_\alpha \to \mathbb{C}$ as

$$\psi_\alpha = \sum_{p\in I} f_{p\alpha}h_p.$$

Note that

$$\psi_\beta - \psi_\alpha = \sum_{p\in I}(f_{p\beta} - f_{p\alpha})h_p = \sum_{p\in I} f_{\alpha\beta}h_p = f_{\alpha\beta}.$$

So if we set $\lambda_\alpha = \exp(2\pi i\psi_\alpha)$ we have that

$$\begin{aligned}
\lambda_\alpha g^1_{\alpha\beta}\lambda_\beta^{-1} &= \exp(2\pi i\psi_\alpha)g^1_{\alpha\beta}\exp(-2\pi i\psi_\beta)\\
&= g^1_{\alpha\beta}\exp(2\pi i f_{\beta\alpha})\\
&= g^1_{\alpha\beta}\exp(\log(g^1_{\beta\alpha}) - \log(g^2_{\beta\alpha}))\\
&= g^1_{\alpha\beta}\exp(\log(g^1_{\beta\alpha}g^2_{\alpha\beta}))\\
&= g^2_{\alpha\beta}.
\end{aligned}$$

Thus, by proposition 1.3.2.2, $L^1 \cong L^2$ and therefore, $[L^1] = [L^2]$ showing the injectivity of κ. □

There are important considerations to be made at this moment. In this category, a line bundle L, over a differentiable manifold M, is trivial if $[L] = [M \times \mathbb{C}]$. This means, in the spirit of proposition 1.3.1.1, admitting a nowhere vanishing section $\sigma : M \to M \times \mathbb{C}$. But looking carefully at this section we realize that there must exist a global function, $f : M \to \mathbb{C}^*$, such that $\sigma(p) = (p, f(p))$ for all $p \in M$. So a trivial object in the category of line bundles, $\check{H}^2(M, \mathbb{Z})$, can be represented as an object in the preceding category, i.e. $\check{H}^1(M, \mathbb{Z})$.

Conversely, a line bundle is not trivial if this does not admit a nowhere vanishing section. In other words, if there is not a global function that trivializes the line bundle. Poincaré's lemma tells us that any closed element in $C^2(\mathcal{U}, \mathbb{Z})$, i.e. $a \in C^2(\mathcal{U}, \mathbb{Z})$ such that $\delta a = 0$, locally is exact, namely $a = \delta b$ for some $b \in C^1(\mathcal{U}, \mathbb{Z})$. Translating this condition into the language of line bundles and global functions we get, on each non empty double intersection $U_{\alpha\beta}$, that

$$g_{\alpha\beta} = f_\beta f_\alpha^{-1}$$

for $f_\alpha : U_\alpha \to \mathbb{C}^*$ and $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$ the transition functions defining a line bundle. What we just did is to reexpress the local triviality of a line bundle from a cohomology perpective. Although, if the line bundle is trivial, $[g] = [\delta f]$, for $f : M \to \mathbb{C}^*$ and here, we are thinking in the languaje of $\check{H}^k(M, C^\infty(\mathbb{C}^*))$.

For instance, if we choose two trivializations for a line bundle described by the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$ we have that

$$g_{\alpha\beta} = f_\beta f_\alpha^{-1} = f'_\beta (f'_\alpha)^{-1},$$

for $f'_\alpha, f_\alpha : U_\alpha \to \mathbb{C}^*$, on $U_{\alpha\beta}$. Then, if we compare these trivializations we obtain

$$\frac{f'_\alpha}{f_\alpha} = \frac{f'_\beta}{f_\beta},$$

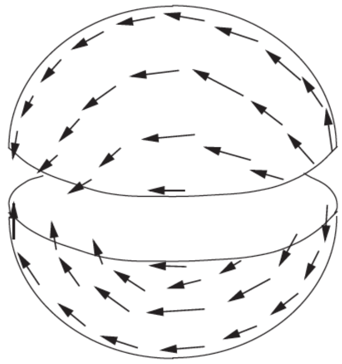
meaning that the the local trivializations on U_α and U_β diverge by a global function. So it made total sense to describe a line bundle by a global functions on non empty double intersections. In the next chapter, when defining a gerbe, we shall see the importance of this point of view.

1.3.3 Examples

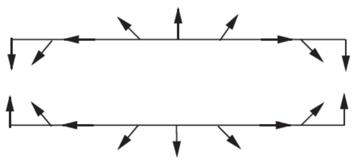
Example 1.3.3.1. ($T\mathbb{S}^2$) Recall that $T\mathbb{S}^2 = \bigsqcup_{p \in \mathbb{S}^2} T_p\mathbb{S}^2$ and that each tangent space can be identified as $T_p\mathbb{S}^2 = \{v \in \mathbb{R}^3 | \langle p, v \rangle = 0\}$. One can endow this space with structure of one dimensional complex vector space taking $(\alpha + i\beta)v = \alpha v + \beta(p \times v)$ so it makes sense to think on $T\mathbb{S}^2$ as a line bundle.

Now, we will provide local trivializations for $T\mathbb{S}^2$ and we do this finding nowhere vanishing sections and its corresponding transition functions just as in remark 1.3.1.2. An atlas for $\mathbb{S}^2$ is given by the upper and lower hemispheres of $\mathbb{S}^2$ but slightly overlapping on the equator in a strip. Let us denote this sets by U_1 and U_2 respectively.

We can find nowhere vanishing sections or smooth vector fields $X_i : U_i \to T\mathbb{S}^2$, for $i = 1, 2$, as in the next figure



One obtains the following figure undoing the strip given by U_{12} and restricting X_1 and X_2 to it



Here, the upper and lower parts correspond to X_1 and X_2 respectively. So if we are asking for the transition functions that solve the equation

$$X_i = g_{ji} X_j,$$

we are finding out how much we have to rotate X_i to get X_j. We express this rotation with the maps $g_{ij} : U_{12} \to \mathbb{C}^*$ with values in the unit circle. Therefore, $T\mathbb{S}^2$ has structure of line bundle.

Example 1.3.3.2. (*The Hopf Bundle*) Recall that $\mathbb{CP}^1$ is the set of all lines through the origin in $\mathbb{C}^2$. We denote the line through the vector $z = (z_1, z_2)$ as $[z] = [z_1, z_2]$. Note that $[\lambda z_1, \lambda z_2] = [z_1, z_2]$ for any $\lambda \in \mathbb{C}^*$.

Now, let $H \subseteq \mathbb{C}^2 \times \mathbb{CP}^1$ be defined as:

$$H = \{(w, [z]) | w = \lambda z \text{ for some } \lambda \in \mathbb{C}^*\}.$$

Define the projection $\pi : H \to \mathbb{CP}^1$ as $(w, [z]) \mapsto [z]$. Hence, the fibers will be given by $H_{[z]} = \pi^{-1}([z]) = \{(\lambda z, [z]) | \lambda \in \mathbb{C}^*\}$. Moreover, we can induce a vector space structure in the fibers setting

$$\lambda_1(w, [z]) + \lambda_2(w', [z]) = (\lambda_1 w + \lambda_2 w', [z]).$$

To verify that H is indeed a line bundle over $\mathbb{CP}^1$ we must find a set of nowhere vanishing sections and its corresponding transition functions as in remark 1.3.1.2. An atlas for $\mathbb{CP}^1$ is given by the open sets $U_i = \{[z_1, z_2] | z_i \neq 0\}$ for $i = 1, 2$ and the coordinate charts $\psi_i = U_i \to \mathbb{C}$ defined as $\psi_1([z_1, z_2]) = z_2/z_1$ and $\psi_2([z_1, z_2]) = z_1/z_2$. Now, consider the sections $\sigma_i : U_i \to H$, for $i = 1, 2$, given by

$$\sigma_1([z]) = ((1, \psi_1([z])), [z])$$

and

$$\sigma_2([z]) = ((\psi_2([z]), 1), [z]).$$

which are clearly nowhere vanishing. Equation (1.3.1.1) tells us that σ_1 and σ_2 are related, on U_{12}, through the transition functions by

$$\sigma_i = g_{ji}\sigma_j.$$

Thus, for $[z] = [z_1, z_2] \in U_{12} \subset \mathbb{CP}^1$:

$$((1, \frac{z_2}{z_1}), [z]) = ((1, \psi_1([z])), [z]) = g_{21}((\psi_2([z]), 1), [z]) = g_{21}((\frac{z_1}{z_2}, 1), [z]).$$

Therefore the transition functions, $g_{ij} : U_{12} \to \mathbb{C}^*$, are

$$g_{21}([z]) = \frac{z_2}{z_1}$$

and

$$g_{12}([z]) = \frac{z_1}{z_2}$$

concluding that H is in fact a line bundle over $\mathbb{CP}^1$ called *the Hopf Bundle.*

Example 1.3.3.3. (*Line Bundles over* $\mathbb{S}^1$, $\mathbb{S}^2$ *and* $\mathbb{S}^3$) Following the ideas from example 1.2.1 we may conclude that the only possible existing line bundles over $\mathbb{S}^1$ and $\mathbb{S}^3$ are the trivial ones, $\mathbb{S}^1 \times \mathbb{C}$ and $\mathbb{S}^3 \times \mathbb{C}$ respectively, up to isomorphism.

So the fact of $\check{H}^2(\mathbb{S}^2, \mathbb{Z})$ being isomorphic to $\mathbb{Z}$ implies that there will be a line bundle over $\mathbb{S}^2$ for each integer. What is more, since $\mathbb{Z}$ is cyclic, there is a unique line bundle over $\mathbb{S}^2$ that generates all the others. In view of $\mathbb{CP}^1 \cong \mathbb{S}^2$, we can think on the Hopf Bundle as a line bundle over $\mathbb{S}^2$, which we denote by L_H. The generator line bundle for $\mathrm{Pic}(\mathbb{S}^2)$ turns out to be L_H [Row].

Chapter 2

The Geometry of Line Bundles

The structure of $\mathbb{R}^n$ gives us a way to differentiate vector fields, which are sections of the tangent bundle over $\mathbb{R}^n$ itself, through the directional derivative. We would like to do the same for line bundles, i.e. differentiating sections of a line bundle, in directions given by vector fields, to obtain sections of it. But there is an obstruction since each fiber of the bundle, unlike $\mathbb{R}^n$, is a different vector space. Actually, there is no canonical way to do that, and a connection is a rule for differentiating sections of the line bundle despite its configuration. Connections give rise through a second derivative, to the notion of curvature for line bundles, and in terms of it we can define toplogical invariants which can be used to classify line bundles.

2.1 Connections and Curvature

Given a smooth manifold M, we will denote the algebra of all smooth complex vector fields on M as $\mathfrak{X}(M)$, the graded algebra of all complex valued smooth differential forms on M as $\Omega(M)$ and the set of all complex valued smooth functions, $f : M \to \mathbb{C}$, as $C^\infty(M)$.

Definition 2.1.1. Let L be a line bundle over a smooth manifold M. A *connection* in L is a $\mathbb{C}$-linear map $\nabla : \mathfrak{X}(M) \to \operatorname{End}\Gamma(M, L)$ defined as $\xi \mapsto \nabla_\xi$ such that for all $\phi \in C^\infty(M)$ and $\sigma \in \Gamma(M, L)$:

- $\nabla_{\phi\xi} = \phi\nabla_\xi$.
- (*Leibniz Rule*) $\nabla_\xi \phi\sigma = (\xi\phi)\sigma + \phi\nabla_\xi\sigma$

Remark 2.1.1. From the first condition in the above definition we note that $(\nabla_\xi\sigma)(p)$, for $p \in M$, depends only on σ and the tangent vector $\xi(p)$ allowing us to define $\nabla_v\sigma$ for any tangent vector v and any section σ. On the other hand, the second condition implies that $(\nabla_\xi\sigma)(p)$ only depends on the germ of σ at p. Thus, if U is any open set, the connection ∇ induces a connection in $L|_U = \pi^{-1}(U)$ and therefore $\nabla_\xi\sigma \in \Gamma(U, L|_U)$ is defined for all $\xi \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(M, L)$.

Now, observe that if $U \subseteq M$ is an open set and $\sigma \in \Gamma(U, L|_U)$ is a nowhere vanishing section we may associate to σ a 1-form $\omega_\sigma \in \Omega^1(U)$ as follows. If $\tau \in \Gamma(U, L|_U)$ is arbitrary, there exists a unique smooth function, $\tau/\sigma \in C^\infty(U)$, such that $\tau = (\tau/\sigma)\sigma$. Then, the map $\mathfrak{X}(U) \to C^\infty(U)$ given by

$$\xi \mapsto \frac{1}{2\pi i}\frac{\nabla_\xi \sigma}{\sigma}$$

is clearly $\mathbb{C}$-linear and a $C^\infty(U)$-module homomorphism. But, $\Omega^1(U) \cong \mathrm{Hom}_{C^\infty(U)}(\mathfrak{X}(U), C^\infty(U))$ which implies that there exists a unique 1-form $\omega_\sigma \in \Omega^1(U)$ such that

$$\nabla_\xi \sigma = 2\pi i \omega_\sigma(\xi)\sigma. \tag{2.1.0.1}$$

Proposition 2.1.1. Let L be a line bundle over a smooth manifold M with connection ∇, $U \subseteq M$ an open set and $\sigma, \tau \in \Gamma^*(U, L|_U)$ which is the set of all nowhere vanishing smooth sections over U. Then we have that

$$\omega_\tau = \omega_\sigma + \frac{1}{2\pi i}\frac{dg}{g}$$

where $g = \tau/\sigma$.

Proof. Let $\xi \in \mathfrak{X}(M)$. By equation (2.1.0.1) we have that

$$2\pi i \omega_\tau(\xi) = \frac{\nabla_\xi \tau}{\tau} = \frac{\nabla_\xi (g\sigma)}{g\sigma}.$$

On the other hand, by the Leibniz Rule, $\nabla_\xi(g\sigma) = (\xi g)\sigma + g\nabla_\xi\sigma$ which implies that

$$\begin{aligned}
2\pi i \omega_\tau(\xi) &= \frac{\nabla_\xi (g\sigma)}{g\sigma} = \frac{(\xi g)\sigma + g\nabla_\xi \sigma}{g\sigma} \\
&= \frac{\xi g}{g} + \frac{\nabla_\xi \sigma}{\sigma} = \frac{dg(\xi)}{g} + \frac{\nabla_\xi \sigma}{\sigma} \\
&= \frac{dg(\xi)}{g} + 2\pi i \omega_\sigma(\xi).
\end{aligned}$$

This yields the desired equality since ξ was arbitrary. □

The next result, which is a corollary to proposition 2.1.1, consist in applying what we just prove to the set of nowhere vanishing sections of a line bundle and its corresponding transition functions.

Corollary 2.1.1.1. Let $g_{\alpha\beta}$ be the corresponding transition functions for the line bundle L relative to the set $\{(\sigma_\alpha, U_\alpha)\}_{\alpha\in I}$ of nowhere vanishing sections. If ∇ is a connection in L and $\omega_\alpha = \omega_{\sigma_\alpha} \in \Omega^1(U_\alpha)$ then one has on $U_{\alpha\beta}$:

$$\omega_\beta = \omega_\alpha + \frac{1}{2\pi i}\frac{dg_{\alpha\beta}}{g_{\alpha\beta}}. \tag{2.1.0.2}$$

Conversely, if a family of 1-forms $\omega_\alpha \in \Omega^1(U_\alpha)$ satisfy the above relation, then there is a unique connection, ∇, such that $\omega_\alpha = \omega_{\sigma_\alpha}$.

Proof. Recall, by equation (1.3.1.1), that whenever $U_{\alpha\beta} \neq \varnothing$ one has $g_{\alpha\beta}\sigma_\alpha = \sigma_\beta$. So the first result follows immediately by proposition 2.1.1 setting $\tau = \sigma_\beta$, $\sigma = \sigma_\alpha$ and $g = g_{\alpha\beta}$.

On the other hand, if we have a set of 1-forms $\omega_\alpha \in \Omega^1(U_\alpha)$ satisfying the equation (2.1.0.2) we define ∇ piecewise as

$$\nabla_\xi \sigma = \left(\xi\left(\frac{\sigma}{\sigma_\alpha}\right) + \frac{\sigma}{\sigma_\alpha} 2\pi i \omega_\alpha(\xi)\right)\sigma_\alpha$$

on U_α for all $\xi \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(M, L)$. Note that on $U_{\alpha\beta}$ one has

$$\begin{aligned}\left(\xi\left(\frac{\sigma}{\sigma_\beta}\right) + \frac{\sigma}{\sigma_\beta} 2\pi i \omega_\beta(\xi)\right)\sigma_\beta &= \left(\xi\left(\frac{\sigma}{\sigma_\beta}\right) + \frac{\sigma}{\sigma_\beta} 2\pi i\left(\omega_\alpha(\xi) + \frac{1}{2\pi i}\frac{dg_{\alpha\beta}(\xi)}{g_{\alpha\beta}}\right)\right)g_{\alpha\beta}\sigma_\alpha \\ &= \left(g_{\alpha\beta}\xi\left(\frac{\sigma}{\sigma_\beta}\right) + \frac{\sigma}{\sigma_\alpha}2\pi i \omega_\alpha(\xi) + \frac{\sigma}{\sigma_\alpha}\frac{dg_{\alpha\beta}(\xi)}{g_{\alpha\beta}}\right)\sigma_\alpha.\end{aligned}$$

In addition, by the Leibniz rule,

$$\xi\left(\frac{\sigma}{\sigma_\alpha}\right) = \xi\left(g_{\alpha\beta}\frac{\sigma}{\sigma_\beta}\right) = g_{\alpha\beta}\xi\left(\frac{\sigma}{\sigma_\beta}\right) + \frac{\sigma}{\sigma_\beta}\xi(g_{\alpha\beta}) = g_{\alpha\beta}\xi\left(\frac{\sigma}{\sigma_\beta}\right) + \frac{\sigma}{\sigma_\alpha}\frac{dg_{\alpha\beta}(\xi)}{g_{\alpha\beta}}.$$

Combining both equalities we obtain

$$\nabla_\xi \sigma = \left(\xi\left(\frac{\sigma}{\sigma_\alpha}\right) + \frac{\sigma}{\sigma_\alpha}2\pi i \omega_\alpha(\xi)\right)\sigma_\alpha = \left(\xi\left(\frac{\sigma}{\sigma_\beta}\right) + \frac{\sigma}{\sigma_\beta}2\pi i \omega_\beta(\xi)\right)\sigma_\beta$$

on $U_{\alpha\beta}$ implying that ∇ is well defined.

Now, we will check that ∇ satisfies both conditions stated in definition 2.1.1 showing that it is indeed a connection. Let $\phi \in C^\infty(M)$ and $\sigma \in \Gamma(M, L)$. First, note that

$$\begin{aligned}\nabla_{\phi\xi}\sigma &= \left(\phi\xi\left(\frac{\sigma}{\sigma_\alpha}\right) + \frac{\sigma}{\sigma_\alpha}2\pi i \omega_\alpha(\phi\xi)\right)\sigma_\alpha \\ &= \left(\phi\xi\left(\frac{\sigma}{\sigma_\alpha}\right) + \frac{\sigma}{\sigma_\alpha}2\pi i \phi\omega_\alpha(\xi)\right)\sigma_\alpha \\ &= \phi\left(\xi\left(\frac{\sigma}{\sigma_\alpha}\right) + \frac{\sigma}{\sigma_\alpha}2\pi i \omega_\alpha(\xi)\right)\sigma_\alpha \\ &= \phi\nabla_\xi\sigma.\end{aligned}$$

On the other hand,

$$\begin{aligned}\nabla_\xi(\phi\sigma) &= \left(\xi\left(\frac{\phi\sigma}{\sigma_\alpha}\right) + \frac{\phi\sigma}{\sigma_\alpha}2\pi i \omega_\alpha(\xi)\right)\sigma_\alpha \\ &= \left(\phi\xi\left(\frac{\sigma}{\sigma_\alpha}\right) + \frac{\sigma}{\sigma_\alpha}\xi(\phi) + \phi\frac{\sigma}{\sigma_\alpha}2\pi i \omega_\alpha(\xi)\right)\sigma_\alpha \\ &= \sigma\xi(\phi) + \left(\phi\xi\left(\frac{\sigma}{\sigma_\alpha}\right) + \phi\frac{\sigma}{\sigma_\alpha}2\pi i \omega_i(\xi)\right)\sigma_\alpha \\ &= \sigma\xi(\phi) + \phi\nabla_\xi\sigma.\end{aligned}$$

Finally, it is clear that $\omega_{\sigma_\alpha} = \omega_\alpha$ which implies that ∇ is unique. $\square$

Remark 2.1.2. The last corollary and equation (2.1.0.1) tell us that we can understand a connection, locally, as a 1-form that glues together properly on twofold intersections through the equation (2.1.0.2).

If we fix a connection ∇ over a line bundle and take the exterior derivative on both sides of equation (2.1.0.2) we get that

$$\begin{aligned} d\omega_\beta &= d\left(\omega_\alpha + \frac{1}{2\pi i}\frac{dg_{\alpha\beta}}{g_{\alpha\beta}}\right) \\ &= d\omega_\alpha + \frac{1}{2\pi i} d\left(\frac{dg_{\alpha\beta}}{g_{\alpha\beta}}\right) \\ &= d\omega_\alpha + \frac{1}{2\pi i}(d(g_{\alpha\beta}^{-1}) \wedge dg_{\alpha\beta} + g_{\alpha\beta}^{-1} \wedge d^2 g_{\alpha\beta}) \\ &= d\omega_\alpha + \frac{1}{2\pi i}\frac{1}{g_{\alpha\beta}^2} dg_{\alpha\beta} \wedge dg_{\alpha\beta} \\ &= d\omega_\alpha \end{aligned}$$

whenever $U_{\alpha\beta} \neq \varnothing$. So there is a global 2-form, F, such that $F|_{U_\alpha} = d\omega_\alpha$.

Definition 2.1.2. Let L be a line bundle with connection ∇. The 2-form F is called the *curvature* of ∇.

Curvature allow us to define topological invariants for line bundles but before doing so we require to discuss some facts about it.

Proposition 2.1.2. The curvature F of a connection ∇ satisfies:

- $dF = 0$.
- If ∇, ∇' are two connections, then $\nabla = \nabla' + \eta$ for η a 1-form and $F_\nabla = F_{\nabla'} + d\eta$.
- If Σ is a closed surface then
$$\frac{1}{2\pi i}\int_\Sigma F_\nabla$$
is an integer independent of ∇.

Definition 2.1.3. Let L be a line bundle over a closed surface Σ. The *chern class* of L, $c(L)$, is defined to be the integer

$$\frac{1}{2\pi i}\int_\Sigma F_\nabla$$

for any connection ∇ over L.

Therefore, the chern class is a way of classifying line bundles topologically. So if for instance, two line bundles are isomorphic, one would expect that its corresponding chern classes are equal.

Remark 2.1.3. What is the analogous of the chern class for global functions? Well, instead of integrating a 2-form over a closed surface, we would need to integrate a 1-form over a closed curve. More concretely, given a closed curve γ and $f : \gamma \to \mathbb{C}^*$ a global function,

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{f(z)}df$$

is an integer known as the *winding number* of f which, as we expect, classifies topologically a global function.

2.2 Examples

Example 2.2.1. ($T\mathbb{S}^2$) We already know that $T\mathbb{S}^2$ has structure of line bundle so our next step will be to construct a connection, $\nabla : \mathfrak{X}(\mathbb{S}^2) \to \mathrm{End}(\mathfrak{X}(\mathbb{S}^2))$, on it and to do so we will follow [Tu17]. If we think on $\mathbb{S}^2$ as a submanifold of $\mathbb{R}^3$ the relation

$$T_p\mathbb{R}^3 = T_p\mathbb{S}^2 \oplus \mathrm{Span}\{v_p\}$$

holds for a normal vector v_p to $T_p\mathbb{S}^2$. Let $\mathrm{pr}_p : T_p\mathbb{R}^3 \to T_p\mathbb{S}^2$ be the corresponding projection. If $X \in \mathfrak{X}(\mathbb{R}^3)$, then $X_p \in T_p\mathbb{R}^3$. Thus, we can define a smooth vector field, $\mathrm{pr}(X)$, on $\mathbb{S}^2$ by

$$(\mathrm{pr}(X))_p = \mathrm{pr}_p(X_p) \in T_p\mathbb{S}^2.$$

Let $X, Y \in \mathfrak{X}(\mathbb{S}^2)$. In general, the directional derivative, $D_{X_p}Y$, for some $p \in \mathbb{S}^2$, is not tangent to $\mathbb{S}^2$ so we define the connection ∇ as

$$(\nabla_X Y)_p = \nabla_{X_p} Y = \mathrm{pr}_p(D_{X_p}Y).$$

Now that we know how to build a connection over $\mathbb{S}^2$ we are able to make an explicit calculation. Choose polar coordinates over $\mathbb{S}^2$ and an appropriate set of charts. If we fix a coordinate chart, the tangent vectors at each point turn out to be

$$\frac{\partial}{\partial\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

and

$$\frac{\partial}{\partial\varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0).$$

Let

$$\vec{n} = \frac{\frac{\partial}{\partial\theta}\times\frac{\partial}{\partial\varphi}}{\|\frac{\partial}{\partial\theta}\times\frac{\partial}{\partial\varphi}\|} = \frac{(\sin^2\theta\cos\varphi, \sin^2\theta\sin\varphi, \sin\theta\cos\theta)}{\sin\theta} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

be a normal vector to each tangent space of $\mathbb{S}^2$, $X \in \mathfrak{X}(\mathbb{S}^2)$ an arbitrary vector field and

$$Y = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} = (-\sin\varphi, \cos\varphi, 0) \in \mathfrak{X}(\mathbb{S}^2).$$

Hence,

$$\nabla_X Y = D_X Y - \langle \vec{n}, D_X Y\rangle \vec{n} = \cos\theta(\vec{n} \times Y)d\varphi(X).$$

Now, recall that on $T\mathbb{S}^2$ we establish that $(\alpha + i\beta)v = \alpha v + \beta(p \times v)$ for $v \in T_p\mathbb{S}^2$ meaning that

$$\nabla_X Y = i\cos\theta Y d\varphi(X).$$

So locally, the connection ∇ is given by the 1-form $\omega = i\cos\theta d\varphi$ and the curvature will be the global 2-form $F = i\sin\theta d\varphi \wedge d\theta$. But the volume form for the two dimensional sphere is given by $\mathrm{vol} = -\sin\theta d\varphi \wedge d\theta$, therefore, $F = -i\mathrm{vol}$.

We conclude this example calculating the chern class of $T\mathbb{S}^2$. Following definition 2.1.3 we get that

$$\begin{aligned} c(T\mathbb{S}^2) &= \frac{1}{2\pi i}\int_{\mathbb{S}^2} F \\ &= \frac{1}{2\pi i}\int_{\mathbb{S}^2} -i\mathrm{vol} = -\frac{4\pi i}{2\pi i} = -2. \end{aligned}$$

Chapter 3

Locally Trivialized Gerbes

We have seen in the first chapter that both complex-valued functions and line bundles, on a smooth manifold M, can be seen as geometric representations of cohomology classes in $\check{H}^0(M, C^\infty(\mathbb{C}^*))$ and $\check{H}^1(M, C^\infty(\mathbb{C}^*))$, respectively. In this chapter we will study a geometric representation for an object in the next degree in cohomology and, using the tools described in chapter 2, how to give sense to the "geometry" of this objects often called *gerbes*.

3.1 What is a Gerbe?

Let M be a smooth manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ a contractible open cover of M. Following the lines of [Cha98] we will define a *gerbe* over M as a geometric object $\mathcal{G}$ which realizes a cohomology class $[g] \in \check{H}^2(M, C^\infty(\mathbb{C}^*))$. However, the parallelism with the case of cohomology classes in $\check{H}^1(M, C^\infty(\mathbb{C}^*)) \cong \check{H}^2(M, \mathbb{Z})$ is not complete since there is no *global* object associated to it (like line bundles are associated to classes in $\check{H}^1(M, C^\infty(\mathbb{C}^*)) \cong H^2(M, \mathbb{Z})$). Instead, as we will see, we will have a family of objects associated to the covering which, coherently considered on the manifold (with respect to the cohomological information) give rise to the gerbe.

Let us start from $[g] \in \check{H}^2(M, C^\infty(\mathbb{C}^*))$, this means that for all $\alpha, \beta, \gamma \in I$ such that $U_{\alpha\beta\gamma} \neq \varnothing$, $g_{\alpha\beta\gamma}$ is a $\mathbb{C}^*$-valued smooth map and $g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma} = g^{-1}_{\alpha\gamma\beta} = g^{-1}_{\gamma\beta\alpha}$. Also, this collection of maps must satisfy the following cocycle condition

$$g_{\beta\gamma\delta} g^{-1}_{\alpha\gamma\delta} g_{\alpha\beta\delta} g^{-1}_{\alpha\beta\gamma} = 1$$

on $U_{\alpha\beta\gamma\delta}$. We say that we trivialize the cocycle $[g]$ with respect to the covering $\mathcal{U}$ if we have a collection $f_{\alpha\beta} = f^{-1}_{\beta\alpha} : U_{\alpha\beta} \to \mathbb{C}^*$ such that

$$g_{\alpha\beta\gamma} = f_{\beta\gamma} f^{-1}_{\alpha\gamma} f_{\alpha\beta}$$

and this is of course equivalent to say that $g_{\alpha\beta\gamma} = (\delta f)_{\alpha\beta\gamma}$ for $f \in C^1(\mathcal{U}, C^\infty(\mathbb{C}^*))$. Two different representations of g in this form, say by δf and $\delta f'$, give rise to a line bundle over M as follows. If

$$g_{\alpha\beta\gamma} = f_{\beta\gamma} f_{\alpha\gamma}^{-1} f_{\alpha\beta} = f'_{\beta\gamma}(f'_{\alpha\gamma})^{-1} f'_{\alpha\beta},$$

and we set $h_{\alpha\beta} = f'_{\alpha\beta}/f_{\alpha\beta}$, we will have that

$$h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta} = 1,$$

which, as we saw in the last chapter, defines a line bundle over M. Now, it is clear that on any open set in the covering we can trivialize in this sense $[g] \in \check{H}^2(M, C^\infty(\mathbb{C}^*))$. So if we consider now two open sets U_α and U_β with non-trivial intersection $U_{\alpha\beta}$, we will have on each such open set one trivialization giving rise to a line bundle $L_{\alpha\beta}$ defined on the intersection $U_{\alpha\beta}$. This collection of line bundles prompt the following working definition of what a gerbe over M must be.

Definition 3.1.1. Let M be a manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ an open cover of M. A *locally trivialized gerbe* $\mathcal{G}$ over M is defined by the following data:

- A line bundle $L_{\alpha\beta}$, on every $U_{\alpha\beta} \neq \varnothing$, such that $L_{\alpha\beta} \cong L_{\beta\alpha}^{-1}$.
- For all $\alpha, \beta, \gamma \in I$ such that $U_{\alpha\beta\gamma} \neq \varnothing$, a nowhere vanishing section $\theta_{\alpha\beta\gamma} \in \Gamma(U_{\alpha\beta\gamma}, L_{\beta\gamma} \otimes L_{\alpha\gamma}^{-1} \otimes L_{\alpha\beta})$ such that $\theta_{\alpha\beta\gamma} = \theta_{\beta\alpha\gamma}^{-1} = \theta_{\alpha\gamma\beta}^{-1} = \theta_{\gamma\beta\alpha}^{-1}$.
- $\theta_{\beta\gamma\delta} \otimes \theta_{\alpha\gamma\delta}^{-1} \otimes \theta_{\alpha\beta\delta} \otimes \theta_{\alpha\beta\gamma}^{-1} = 1$ on $U_{\alpha\beta\gamma\delta}$.

Thus, we can associate to each $\theta_{\alpha\beta\gamma}$ a map $g_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to \mathbb{C}^*$ giving rise to a class $[g] \in \check{H}^2(M, C^\infty(\mathbb{C}^*))$. Actually, in [Cha98] a notion of equivalence of locally trivialized gerbes is given in such a way that, as we expect, the following theorem holds.

Theorem 3.1.1. Let M be a smooth manifold. The collection of equivalence classes of locally trivialized gerbes is isomoporhic to

$$\check{H}^2(M, C^\infty(\mathbb{C}^*)) \cong \check{H}^3(M, \mathbb{Z}).$$

It is worth to be mentioned that the group operation taken over the set of equivalence classes of locally trivialized gerbes is the tensor product. This product is obtained by tensoring the pairs of line bundles and the nowhere vanishing sections in the natural way.

Remark 3.1.1. In general, in the existing literature on "gerbes", they are described as geometric representations of what a cohomology class in $\check{H}^2(M, C^\infty(\mathbb{C}^*))$ is. Thus, there are different approaches to build up such a geometric representation for a gerbe. The original one was given by Giraud in [Gir71] in terms of non-abelian cohomology objects (called "faisceaux", "torseurs", etc.), then there is the one used by Brylinski [Bry08] (in terms of groupoids and stacks), during the 90's the one of *bundle gerbes* used by the australian school (M. Murray, A. Carey and co-workers) and the one we follow (N. Hitchin, D. Chaterjee and others). Notice, however, that the object we call a locally trivialized gerbe in this work is also called a *gerb* in Chatterjee's Ph.D. thesis [Cha98].

3.2 Examples

Although the definition of locally trivialized gerbes (or simply gerbes) looks quite involved, in the following examples we will present some particular instances in which the construction of these objects can be simplified. Indeed, in particular cases, we can use the Hopf fibration (the generator of the Picard group on the two dimensional sphere) and appropriate coverings or the 3-sphere to build very basic gerbes as we are about to explain.

Example 3.2.1. (*Gerbes over* $\mathbb{S}^1, \mathbb{S}^2$ *and* $\mathbb{S}^3$) Here, we extend what we did in example 1.2.1 to the context of gerbes. Since $\check{H}^3(\mathbb{S}^1, \mathbb{Z})$ and $\check{H}^3(\mathbb{S}^2, \mathbb{Z})$ are trivial we can conclude that the only possible existing gerbes over $\mathbb{S}^1$ and $\mathbb{S}^2$ are the trivial ones up to isomorphism. However, since $\check{H}^2(\mathbb{S}^2, \mathbb{Z}) \cong \check{H}^3(\mathbb{S}^3, \mathbb{Z}) \cong \mathbb{Z}$ there will be a gerbe over $\mathbb{S}^3$ for each integer, and we can use what we already know about line bundles over $\mathbb{S}^2$ to induce gerbes over $\mathbb{S}^3$ as follows.

First, we construct a convenient open cover for $\mathbb{S}^3$ and, to do so, choose $p \in \mathbb{S}^3$ and $U_p \subseteq \mathbb{S}^3$ an open set containing p such that U_p is diffeomorphic to $\mathbb{R}^3$. Now, let $U_0 = \mathbb{S}^3 \setminus \{p\}$, then clearly the set $\mathcal{U} = \{U_p, U_0\}$ is an open cover for $\mathbb{S}^3$. We need now to bring line bundles over $\mathbb{S}^3$ from those we know to exist over $\mathbb{S}^2$.

Let us recall the definition of pull-back bundle. Let M, M' be differentiable manifolds, a line bundle L over M with projection $\pi : L \to M$ and $f : M' \to M$ a C^∞ map. The *pull-back bundle* over M' is defined to be

$$f^*L = \{(p', q) \in M' \times L | f(p') = \pi(q)\}$$

endowed with the subspace topology. Now, observe that $U_{0p} \cong \mathbb{R}^3 \setminus \{0\} \cong \mathbb{S}^2 \times \mathbb{R}$. Let $\pi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2$ be the canonical projection and L_H be the Hopf Bundle over $\mathbb{S}^2$. With this in mind, we can produce a line bundle over U_{0p} setting $L_{0p} := \pi^* L_H$. This allows us to define a gerbe $\mathcal{G}_H$ over $\mathbb{S}^3$ taking L_{0p} on $U_0 \cap U_p$ and, given that $\mathcal{U}$ has only two open sets, we do not need to care about triple intersections. Notice that this gerbe is not a line bundle over $\mathbb{S}^3$, because it is not defined as a "global" line bundle over the full of the three dimensional sphere. As a matter of fact, the gerbe $\mathcal{G}_H$ is the generator of $\check{H}^3(\mathbb{S}^3, \mathbb{Z}) \cong \mathbb{Z}$ [Hit01].

Example 3.2.2. (*A gerbe over compact connected Lie group*) Given a compact connected finite-dimensional Lie group G, since $\check{H}^3(G, \mathbb{Z}) \cong \mathbb{Z}$ [DK00], there is a gerbe over G for each integer n. Indeed, the de Rham representative for the third cohomology of G can be written down in terms of two basic operations on the Lie algebra $\mathfrak{g}$ of G. Let us denote by $[X, Y]$ the Lie bracket of two elements $X, Y \in \mathfrak{g}$, and $\langle , \rangle_{\mathfrak{g}}$ be a non-degenerate symmetric bilinear form on $\mathfrak{g}$. Then, the expression

$$H_G = \frac{1}{2}\langle [X, Y], Z\rangle_{\mathfrak{g}},$$

defines non-trivial 3-form on G. (This 3-form can be written, using the notation $g \in G$ for elements in the Lie group G, as (up to a constant) $H_G = \frac{1}{2}\mathrm{tr}(g^{-1}dg)^3$.)

3.3 The Geometry of Gerbes

Let M be a smooth manifold, $\mathcal{U} = \{U_i\}_{i \in I}$ a contractible open cover of M and L a line bundle over M. Recall that a connection on L is a one form ω_α on U_α, for all $\alpha \in I$, such that equation (2.1.0.2) holds on $U_{\alpha\beta}$. Also, remember that if we take the exterior derivative on both sides of this equation we get $d\omega_\alpha = d\omega_\beta$ so that there exists a global 2-form, F, such that $F|_{U_\alpha} = d\omega_\alpha$ called the curvature of the connection.

There is another approach to do this and is by relating the de Rham to the Čech cohomology. Suppose that F is a closed 2-form on M. By the Poincaré lemma there exist 1-forms ω_α defined on U_α for all $\alpha \in I$ such that $F|_{U_\alpha} = d\omega_\alpha$. But, $d\omega_\alpha - d\omega_\beta = d(\omega_\alpha - \omega_\beta) = 0$ on $U_{\alpha\beta}$ so again, by the Poincaré Lemma,

$$\omega_\alpha - \omega_\beta = df_{\alpha\beta}$$

for some $f_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{R}$. But by the equivalence between the de Rham and Čech cohomologies one may choose $f_{\alpha\beta}$ so that $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$, given by $g_{\alpha\beta} = \exp(2\pi i f_{\alpha\beta})$, is a cocycle, i.e. the transition function describing a line bundle. Solving this equality for $f_{\alpha\beta}$ one gets

$$f_{\alpha\beta} = \frac{1}{2\pi i} \log g_{\alpha\beta}.$$

Hence,

$$\omega_\alpha - \omega_\beta = df_{\alpha\beta} = d\left(\frac{1}{2\pi i} \log g_{\alpha\beta}\right) = \frac{1}{2\pi i} \frac{dg_{\alpha\beta}}{g_{\alpha\beta}}$$

yielding equation (2.1.0.2).

We can replicate the same idea for a gerbe, G, over a differentiable manifold M defined by the cocycles $g_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to \mathbb{C}^*$. Doing so we get that there exist a global closed 3-form, K, a 2-form, F_α, on each $U_\alpha \in \mathcal{U}$ and a 1-form, $\omega_{\alpha\beta}$, on each non empty double intersection $U_{\alpha\beta}$ such that:

$$K|_{U_\alpha} = dF_\alpha$$

for all $\alpha \in I$,

$$F_\beta - F_\alpha = d\omega_{\alpha\beta}$$

on $U_{\alpha\beta}$ and

$$\omega_{\beta\gamma} - \omega_{\alpha\gamma} + \omega_{\alpha\beta} = \frac{1}{2\pi i} \frac{dg_{\alpha\beta\gamma}}{g_{\alpha\beta\gamma}}$$

on $U_{\alpha\beta\gamma}$.

Definition 3.3.1. Let $\mathcal{G}$ be the gerbe, over a differentiable manifold M, defined by the cocycles $g_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to \mathbb{C}^*$. The set of 2-forms F_α for all $\alpha \in I$ is what we define to be a *connection* on G and the global 3-form K is called the *curvature* associated to the connection F.

Remark 3.3.1. The analogous of the chern class and the winding number for gerbes is commonly known as the *Dixmier-Douady class* .

Example 3.3.1. Consider the gerbe $\mathcal{G}_H$ over the three dimensional sphere $\mathbb{S}^3$ constructed in example 3.2.1. In [Hit01] is shown how to realize the volume form $\kappa := d\,\mathrm{vol}_{\mathbb{S}^3}$ as the curvature of a connection on the gerbe $\mathcal{G}_H$. Let us denote by Δ the Laplacian on the 3-sphere, then we can solve the differential equation

$$\Delta H = \kappa$$

on the open set U_p, and we denote the solution by $H_p \in \Omega^3(U_p)$ and the differential equation

$$\Delta H = \kappa + 2\pi\delta_p,$$

where δ_p denotes the Dirac delta distribution, on the open set U_0; we denote the corresponding solution by $H_0 \in \Omega^3(U_0)$. Then, there exist 2-forms $F_0 = d^*H_0$ and $F_p = d^*H_p$ on the corresponding open sets in $\mathbb{S}^3$ such that

$$dF_0 = dd^*H_0 = \Delta H_0 = \kappa = \Delta H_p = dd^*H_p = dF_p$$

on the intersection U_{0p}. It is then possible to show that

$$\frac{1}{2\pi}(F_p - F_0) = \frac{1}{2\pi}d^*(H_p - H_0) \in H^2_{\mathrm{dR}}(U_{0p}, \mathbb{Z}),$$

so that it corresponds to the curvature of a connection on the line bundle L_{0p} over U_{0p}, i.e. a gerbe connection for $\mathcal{G}_H$.

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