The spectral description of the Hitchin fibration

Juan Sebastián Numpaque Roa

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Advisor:

Florent Schaffhauser

Universidad de los Andes

Facultad de Ciencias, Departamento de Matemáticas

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Tu bien sabes, Merceditas, desde donde quiera que estés,
que a ti consagro toda labor y trabajo que hago.

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Contents

Introduction

Higgs bundles emerged in the late 80's in Nigel Hitchin's study of the self-duality equations over a Riemann surface [Hit87a] and in Carlos Simpon's subsequent work on non-abelian Hodge theory [Sim92]. These bundles have a rich geometry and have become relevant in many different areas including symplectic geometry and integrable systems [Hit87b], surface group representations [BGPG03], mirror symmetry [Bra17] and Langlands duality [Hit07].

In the celebrated paper *"Stable bundles and integrable systems"* [Hit87b], Nigel Hitchin showed that one can study the moduli space of stable Higgs bundles of fixed rank r and degree d over a compact Riemann surface M, $\mathcal{M}(r,d)$, through a map from this space to $\bigoplus_{n=1}^{r} H^0(M, K_M^{\otimes n})$ known, today, as the Hitchin Fibration. He went further studying the fiber of this map by introducing and working with spectral curves.

In this thesis we focus on making a clear picture of the previous discussion. The facts presented here are well known but we gave ourselves the task of writing an expository account of these, providing precise references whenever it cannot be self-contained.

The work is divided in two chapters and one appendix. In the first chapter we define Higgs bundles and review basic facts about them, then we see that it is possible to define analogues to the trace, determinant and any Ad-invariant polynomial in the Higgs bundles setting and this will allow us to define and talk about the Hitchin fibration at the end of the chapter. In the second chapter we construct an analogue of the characteristic polynomial for Higgs bundles. The study of the set of zeroes of this characteristic polynomial will require that we introduce spectral curves. By studying these curves we will find out that there are beautiful relations between its geometry and the Higgs bundle that give rise to them. We conclude the chapter linking the fiber of the Hitchin fibration to a very special abelian variety over a given spectral curve. Finally, in the appendix, we briefly introduce concepts that play an important role in the work we have done and that where no studied in due time because that would have diverted our attention from the main discussion.

Chapter 1

Higgs bundles & the Hitchin fibration

The main goal of this chapter is to define the Hitchin Fibration. To do so we will first talk about Higgs bundles and give examples of them. Then, we will see how invariant polynomials relate to sections of the canonical bundle and its powers. All this together will give us the necessary tools to give the promised definition of the Hitchin fibration.

In what follows, M will be a compact and connected Riemann surface with genus $g := \dim(H^0(M, K_M)) > 1$ and $\pi : K_M \to M$ will be its canonical bundle. Recall that for an arbitrary n-dimensional complex manifold, X,

$$K_X = \bigwedge^n \left(T^*X\right)$$

where T^*X is the holomorphic cotangent bundle [GH94, Chapter 1]. The sections of K_X are precisely the holomorphic differential n forms on X. For us, $n = 1$ so K_M is just the holomorphic cotangent bundle of M.

1.1 Higgs bundles

We start with an example that serves as motivation for what is coming next in this section.

Example 1.1. Recall that an element in the Picard group of M, $H^1(M,\mathcal{O}_M^*)$, stands for an equivalence class of isomorphic line bundles over M. From the exact sequence

$$0 \longrightarrow \frac{H^1(M,\mathcal{O}_M)}{H^1(M,\mathbb{Z})} \cong \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}} \longrightarrow H^1(M,\mathcal{O}_M^*) \xrightarrow{\deg} H^2(M,\mathbb{Z}) \cong \mathbb{Z} \longrightarrow 0$$

we can conclude that the set of isomorphism classes of line bundles with fixed degree is a complex torus isomorphic to the Jacobian, $\mathrm{Pic}_0(M)$ ={Isomorphism classes of degree 0 line bundles}, of the Riemann surface M. Since the Jacobian has structure of complex Lie group, its tangent bundle is trivial and isomorphic to $\mathrm{Pic}_0(M) \times H^1(M,\mathcal{O}_M)$. The cotangent bundle will be trivial as well and, by Serre duality, $H^1(M,\mathcal{O}_M) \cong H^0(M,K_M)^*$. Then, given that $\mathrm{End}(L) \cong \mathcal{O}_M$ for any $L \in \mathrm{Pic}_0(M)$,

$$T^*\mathrm{Pic}_0(M) \cong \mathrm{Pic}_0(M) \times H^0(M,K_M) \cong \mathrm{Pic}_0(M) \times H^0(M,\mathrm{End}(L) \otimes K_M).$$

In the last example, any point $(L,\phi) \in T^*\mathrm{Pic}_0(M)$ corresponds to the rank 1 case of the following definition, which is due to to N.Hitchin (Higgs field) and C.Simpson (Higgs bundle) [BGPG07].

Definition 1.2. A *Higgs Bundle* over M is a pair (E,ϕ) where $p: E \to M$ is a holomorphic vector bundle on M and ϕ is a holomorphic 1-form with values in $\mathrm{End}(E)$, that is, $\phi \in H^0(M,\mathrm{End}(E) \otimes K_M)$. The map ϕ is called *Higgs field*.

Remark 1.3. One can think of global sections in $H^0(M,\mathrm{End}(E) \otimes K_M)$ as elements in the vector space $H^0(M,\mathrm{Hom}(E,E \otimes K_M))$ since

$$\mathrm{End}(E) \otimes K_M \cong (E^* \otimes E) \otimes K_M \cong E^* \otimes (E \otimes K_M) \cong \mathrm{Hom}(E,E \otimes K_M).$$

This point of view is useful to make sense of the following examples of Higgs bundles.

Example 1.4. By the square root of a line bundle, L, over M we mean a line bundle $L^{1/2}$ such that $L^{1/2} \otimes L^{1/2} = L$. Notice that $\deg(L) = 2\deg(L^{1/2})$ so if L has a square root then its degree must be even. By the Riemann-Roch theorem we can show that $\deg(K_M) = 2g-2$ and, therefore, it is natural to ask wether the canonical bundle of M admits a square root or not. Indeed, there are 2^{2g} different square roots for K_M [Ati71]. Now, let $K_M^{1/2}$ be a square root for the canonical bundle of M, $\omega \in H^0(M, K_M^{\otimes 2})$, $E = K_M^{1/2} \oplus K_M^{-1/2}$ and ϕ given by

$$\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}.$$

For $\alpha \in H^0(M, K^{1/2})$ and $\beta \in H^0(M, K^{-1/2})$ we have

$$\begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \omega \otimes \beta \\ \alpha \end{pmatrix} \in H^0(M, K_M^{3/2} \oplus K_M^{1/2})$$

with $K_M^{3/2} := K_M^{\otimes 2} \otimes K_M^{-1/2}$. Given that $K_M^{3/2} \oplus K_M^{1/2} \cong E \otimes K_M$, (E, ϕ) can be regarded as a Higgs bundle of rank 2.

Example 1.5. Let $E = K_M \oplus \mathcal{O}_M \oplus K_M^{-1}$ and ϕ given by

$$\phi = \begin{pmatrix} \frac{\nu}{3} & \frac{\mu}{2} & \lambda \\ 1 & \frac{\nu}{3} & \frac{\mu}{2} \\ 0 & 1 & \frac{\nu}{3} \end{pmatrix}$$

with $\nu \in H^0(M, K_M)$, $\mu \in H^0(M, K_M^{\otimes 2})$ and $\lambda \in H^0(M, K_M^{\otimes 3})$. As with the example above, notice that for $(\alpha, \beta, \gamma) \in H^0(M, K_M \oplus \mathcal{O}_M \oplus K_M^{-1})$,

$$\begin{pmatrix} \frac{\nu}{3} & \frac{\mu}{2} & \lambda \\ 1 & \frac{\nu}{3} & \frac{\mu}{2} \\ 0 & 1 & \frac{\nu}{3} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \frac{\nu}{3} \otimes \alpha + \frac{\mu}{2} \otimes \beta + \lambda \otimes \gamma \\ \alpha + \frac{\nu}{3} \otimes \beta + \frac{\mu}{2} \otimes \gamma \\ \beta + \frac{\nu}{3} \otimes \gamma \end{pmatrix} \in H^0(M, K_M^{\otimes 2} \oplus K_M \oplus \mathcal{O}_M).$$

Since $K_M^{\otimes 2} \oplus K_M \oplus \mathcal{O}_M \cong (K_M \oplus \mathcal{O}_M \oplus K_M^{-1}) \otimes K_M$, (E, ϕ) gives us a rank 3 Higgs Bundle.

Of course, we have a notion of morphism between Higgs bundles so that we can think of them as a category.

Definition 1.6. Let $(E_1, \phi_1), (E_2, \phi_2)$ be Higgs bundles over M. A map, $f : (E_1, \phi_1) \to (E_2, \phi_2)$, is a *morphism of Higgs bundles* if it is a vector bundle homomorphism such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\quad f \quad} & E_2 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ E_1 \otimes K_M & \xrightarrow[f \otimes \mathrm{Id}_{K_M}]{} & E_2 \otimes K_M \end{array}$$

commutes.

If we want, as in Example 1.1, a space parametrizing isomorphism classes of Higgs bundles (E,ϕ) with fixed $\mathrm{rank}(E) = r > 1$ and $\deg(E) = d$, we cannot consider all the Higgs bundles. It is necessary to impose a stability condition which we are about to discuss.

Definition 1.7. Let (E,ϕ) be a Higgs bundle over M. A vector sub-bundle F for which $\phi(F) \subseteq F \otimes K_M$ is said to be a *ϕ-invariant sub-bundle* of E. Moreover, if for each proper ϕ-invariant sub-bundle $F \subset E$ one has

$$\mu(F) = \frac{\deg(F)}{\mathrm{rank}(F)} < \frac{\deg(E)}{\mathrm{rank}(E)} = \mu(E),$$

(E,ϕ) is said to be *stable*.

Proposition 1.8. Stability of Higgs bundles has the following properties [Hit92, Section 2]:

1. If f is a holomorphic automorphism of E and (E,ϕ) is stable, then $(E, f^*\phi)$ is also stable.

2. If (E,ϕ) is stable and $\lambda \in \mathbb{C}^*$, then $(E,\lambda\phi)$ is stable.

3. Stability is an open condition, that is, if (E,ϕ) is stable, then for $\tilde{\phi}$ "near to" ϕ, the Higgs bundle $(E,\tilde{\phi})$ is also stable.

Proof. We are going to show the first two properties. For the first one, let $L \hookrightarrow E$ be a $f^*\phi$-invariant proper sub-bundle. Then, by the definition of $f^*\phi$-invariance,

$$f^*\phi(L) = (f^{-1} \otimes \mathrm{Id}) \circ \phi \circ f(L) \subseteq L \otimes K_M.$$

Since f is an automorphism, $f(L)$ is a proper sub-bundle of E such that $\text{rank}(f(L)) = \text{rank}(L)$, $\deg(f(L)) = \deg(L)$ and $\phi(f(L)) \subseteq f(L) \otimes K_M$. Thus, $f(L)$ is ϕ-invariant and, by the stability of (E, ϕ), $\mu(L) = \mu(f(L)) < \mu(E)$.

Finally, let $\lambda \in \mathbb{C}^*$ and $L \hookrightarrow E$ be a $\lambda\phi$-invariant proper sub-bundle. Note that

$$(\lambda\phi)(L) = \phi(\lambda L) = \phi(L) \subseteq L \otimes K_M$$

which means that L is also ϕ-invariant. The second property follows then from the stability of (E, ϕ). $\square$

Theorem 1.9. Let $\mathcal{M}(r, d)$ be the set of all stable Higgs bundles over M with fixed rank, r, and degree, d, up to isomorphism. Then $\mathcal{M}(r, d)$, which is called *the Moduli space of stable Higgs bundles over M*, has structure of smooth quasi-projective scheme of dimension $2(r^2(g-1)+1)$ [Nit91].

1.2 Invariant polynomials & Symmetric differentials

With respect to a trivializing open set $U_j \subseteq M$ there exist holomorphic maps $f_{k,l} : U_j \to \mathbb{C}$, with $1 \leq k, l \leq r := \text{rank}(E)$ such that

$$\phi|_{U_j} = A_j \otimes dz_j = \begin{pmatrix} f_{1,1}(z_j) & \dots & f_{1,r}(z_j) \\ \vdots & \ddots & \vdots \\ f_{r,1}(z_j) & \dots & f_{r,r}(z_j) \end{pmatrix} \otimes dz_j. \tag{1.1}$$

If g_{ij}, h_{ij} are the transition functions for E and K_M respectively, on the overlaps $U_{ij} := U_i \cap U_j$ we must have that $A_i = \text{Ad}(g_{ij})A_j = g_{ij}A_j g_{ij}^{-1}$, $dz_i = h_{ij}dz_j$ and therefore

$$A_i \otimes dz_i = \phi|_{U_i} = (\text{Ad}(g_{ij}) \otimes h_{ij})\phi|_{U_j} = \text{Ad}(g_{ij})A_j \otimes h_{ij}dz_j.$$

Taking trace in Equation 1.1 we get that

$$\operatorname{Tr}(\phi|_{U_j}) = \operatorname{Tr}(A_j \otimes dz_j) = \sum_{k=1}^{r} f_{k,k} \otimes dz_j.$$

On the other hand, on U_{ij},

$$\operatorname{Tr}(\phi|_{U_i}) = \operatorname{Tr}((\operatorname{Ad}(g_{ij}) \otimes h_{ij})\phi|_{U_j}) = \sum_{k=1}^{r} f_{k,k} \otimes h_{ij} dz_j = h_{ij} \operatorname{Tr}(\phi|_{U_j})$$

since the trace is Ad-invariant. Thus, the set of local sections $\{\operatorname{Tr}(\phi|_{U_j}) : U_j \to \mathbb{C}\}$ piece together well and give rise to a global section in $H^0(M, K_M)$ which we call *trace of the Higgs field* and denote by $\operatorname{Tr}(\phi)$. This is a particular case of a more general fact, which we are about to discuss.

Theorem 1.10. Let $p : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$ be an Ad-invariant homogeneous polynomial with $\deg(p) = n$. Then p induces an holomorphic map

$$\begin{aligned} f_p : \mathcal{M}(r,d) &\to H^0(M, K_M^{\otimes n}) \\ (E, \phi) &\mapsto p(\phi) \end{aligned}$$

with $p(\phi)$ defined as $p(\phi)|_{U_j} = p(A_j) \otimes dz_j^{\otimes n}$ in a trivializing open set $U_j \subseteq M$ [Hit87b].

Proof. We verify that $p(\phi)$ is indeed a global section in $H^0(M, K_M^{\otimes n})$. On the overlaps, U_{ij}, we have that

$$p(\phi)|_{U_i} = p(A_i) \otimes dz_i^{\otimes n} = p(\operatorname{Ad}(g_{ij})A_j) \otimes (h_{ij} dz_j)^{\otimes n} = h_{ij}^{\otimes n} p(\phi)|_{U_j}.$$

Since the cocycles for the line bundle $K_M^{\otimes n}$ are given by $h_{ij}^{\otimes n}$, we can conclude that $p(\phi)$ is a well defined global section in $H^0(M, K_M^{\otimes n})$. □

Remark 1.11. Let $(E, \phi), (E', \phi')$ be two isomorphic Higgs bundles with transition functions given by g_{ij}, g'_{ij} respectively. Recall that there exist holomorphic maps $f_i : U_i \to \operatorname{GL}(r, \mathbb{C})$ such that $\varphi_i = f_i \varphi'_i$ for φ_i and φ'_i local trivializations of E and E' respectively.

Then $A_i = \mathrm{Ad}(f_i)A_i'$ and

$$p(\phi)|_{U_i} = p(A_i) \otimes dz_i^{\otimes m} = p(\mathrm{Ad}(f_i)A_i') \otimes dz_i^{\otimes m} = p(\phi')|_{U_i}.$$

This implies that the map f_p introduced in the previous theorem is well defined since it only depends on the isomorphism class to which a stable Higgs bundle belongs.

Remark 1.12. One can actually define Ad-invariant polynomials for any Higgs bundle. However, on Theorem 1.10 we restricted ourselves to the Moduli space $\mathcal{M}(r,d)$ having in mind the definition of the Hitchin fibration, which we will present in the forthcoming section.

There are plenty of invariant polynomials on the Higgs field that we can consider. However, in the next examples we focus in the ones that are relevant and helpful in our understanding of the Hitchin fibration and the spectral curve associated to a Higgs bundle.

Example 1.13. Let $n \in \mathbb{N}$. For any $A \in \mathfrak{gl}(r,\mathbb{C})$, $\mathrm{Tr}(A^n)$ is an Ad-invariant homogeneous polynomial of degree n. Then, by Theorem 1.10, the map $\mathrm{Tr}(\phi^n)$ locally defined as

$$\mathrm{Tr}(\phi^n)|_{U_j} = \mathrm{Tr}(A_j^n) \otimes (dz_j)^{\otimes n}$$

defines a section in $H^0(M, K_M^{\otimes n})$.

Example 1.14. The characteristic polynomial of a matrix $A \in \mathfrak{gl}(r,\mathbb{C})$ can be written, in terms of the Plemelj-Smithies formulas [RS78, Chapter XIII, Theorem 108], as

$$\det(\eta \mathrm{Id} - A) = \sum_{n=0}^{r} (-1)^n \eta^{r-n} q_n(\mathrm{Tr}(A), \dots, \mathrm{Tr}(A^n))$$

with

$$q_n(\mathrm{Tr}(A),\ldots,\mathrm{Tr}(A^n)) = \frac{1}{n!}\det\begin{pmatrix} \mathrm{Tr}(A) & n-1 & 0 & \ldots & 0 \\ \mathrm{Tr}(A^2) & \mathrm{Tr}(A) & n-2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Tr}(A^{n-1}) & \mathrm{Tr}(A^{n-2}) & \mathrm{Tr}(A^{n-3}) & \ldots & 1 \\ \mathrm{Tr}(A^n) & \mathrm{Tr}(A^{n-1}) & \mathrm{Tr}(A^{n-2}) & \ldots & \mathrm{Tr}(A) \end{pmatrix}.$$

The characteristic polynomial and, in particular, the homogeneous polynomials q_n are invariant under conjugation. Thus, $q_n(\mathrm{Tr}(\phi),\ldots,\mathrm{Tr}(\phi^n))$ is a well defined global section in $H^0(M, K_M^{\otimes n})$.

Example 1.15. Suppose that $\mathrm{rank}(E) = 2$. By Theorem 1.10, we can define the section $\det(\phi) \in H^0(M, K_M^{\otimes 2})$ by

$$\det(\phi)|_{U_j} = (f_{1,1}f_{2,2} - f_{2,1}f_{1,2}) \otimes dz_j^{\otimes 2} = \frac{\mathrm{Tr}(\phi)|_{U_j}^2 - \mathrm{Tr}(\phi^2)|_{U_j}}{2} \otimes dz_j^{\otimes 2}.$$

As a matter of fact,

$$\det(\phi) = \frac{\mathrm{Tr}(\phi)^{\otimes 2} - \mathrm{Tr}(\phi^2)}{2} = q_2(\mathrm{Tr}(\phi), \mathrm{Tr}(\phi^2)).$$

What we saw in the last two examples is no coincidence. Using tools from Invariant Theory, one can show that a basis for the vector space of Ad-invariant polynomials in the entries of any matrix $A \in \mathfrak{gl}(r, \mathbb{C})$ is given by the set $\{\mathrm{Tr}(A), \mathrm{Tr}(A^2), \ldots, \mathrm{Tr}(A^r)\}$ [Pro76, Part 1, Theorem 1.3]. Hence, every Ad-invariant homogeneous polynomial in the entries of the Higgs field can be written in terms of the invariant polynomials introduced in Example 1.13.

1.3 The Hitchin fibration

As seen in Example 1.14, the coeficients, p_n, of the characteristic polynomial of a matrix induce well defined sections $p_n(\phi) := q_n(\mathrm{Tr}(\phi),\ldots,\mathrm{Tr}(\phi^n)) \in H^0(M, K^{\otimes n})$ for $n = 1,\ldots,r$.

These sections play an important role in the definition of the characteristic polynomial associated to a Higgs field and the construction of the spectral curve of a Higgs bundle as we will see in the next chapter. This is, precisely, the motivation for considering the holomorphic map

$$\begin{aligned} h : \mathcal{M}(r,d) &\to \mathcal{A}(r,d) := \bigoplus_{n=1}^{r} H^0(M, K_M^{\otimes n}) \\ (E,\phi) &\mapsto (p_1(\phi),\dots,p_r(\phi)) \end{aligned} \tag{1.2}$$

which is known as the *Hitchin Fibration*. The codomain of this map, $\mathcal{A}(r,d)$, is called the *Hitchin base* and we can calculate its dimension as follows.

Proposition 1.16.

$$\dim(\mathcal{A}(r,d)) = \frac{\dim(\mathcal{M}(r,d))}{2}.$$

Proof. For all $n = 2,\dots,r$ we have, by the Riemann-Roch theorem, that

$$\dim(H^0(M,K_M^{\otimes n})) - \dim(H^1(M,K_M^{\otimes n})) = \deg(K_M^{\otimes n}) + 1 - g = n(2g-2) + 1 - g.$$

On the other hand, by Serre duality,

$$H^1(M,K_M^{\otimes n}) = H^0(M,K_M^{\otimes 1-n})^*.$$

But, since $g > 1$, $\deg(K^{\otimes 1-n}) = (1-n)(2g-2) < 0$ which means that $K_M^{\otimes 1-n}$ does not admit non-zero holomorphic sections and, therefore, $\dim(H^0(M,K_M^{\otimes 1-n})) = 0$. Thus

$$\dim(H^0(M,K_M^{\otimes n})) = (2n-1)(g-1).$$

Finally, if we add all these dimensions we get

$$\sum_{n=1}^{r} \dim(H^0(M,K_M^{\otimes n})) = g + 3(g-1) + \ldots + (2r-1)(g-1) = r^2(g-1) + 1 = \frac{\dim(\mathcal{M}(r,d))}{2}.$$

□

Remark 1.17. One can also define the Hitchin Fibration as the holomorphic map sending (E,ϕ) to $(\mathrm{Tr}(\phi),\dots,\mathrm{Tr}(\phi^k))$ or any generating family for the Ad-invariant polynomials in the entries of the Higgs field ϕ. In some situations, this point of view on the Hitchin fibration is more useful, see for instance [Hit90].

Remark 1.18. We have just constructed the Hitchin Fibration by considering polynomials invariant under the action of $\mathrm{GL}(r,\mathbb{C})$ on its Lie algebra, $\mathfrak{gl}(r,\mathbb{C})$. However, one can also consider invariant polynomials under the action of a Lie group $G \subseteq \mathrm{GL}(n,\mathbb{C})$ on its Lie algebra and study the corresponding Hitchin Fibration. This was done by Hitchin for the classical complex semisimple Lie groups $\mathrm{SL}(r,\mathbb{C}), \mathrm{Sp}(r,\mathbb{C}), \mathrm{SO}(2r+1,\mathbb{C})$ and $\mathrm{SO}(2r,\mathbb{C})$ [Hit87b].

Remark 1.19. One can study the Hitchin Fibration on a more general setting in which one consider pairs (E,ϕ) with $\phi \in H^0(M,\mathrm{End}(E)\otimes L)$ and L an arbitrary line bundle. This is a work by Beauville, Narasimhan and Ramanan [BNR89].

Example 1.20. We will show that the Higgs bundle from Example 1.4 is stable and then, we will give its image under the Hitchin fibration. To do so, we follow the lines of Hitchin [Hit92, Section 3].

We start by considering the Higgs field given by

$$\varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and showing that the corresponding Higgs bundle (E,φ) is stable. Any sub-bundle $L \subset E$ is given locally, on a trivializing open set $U \subset M$, by a nowhere vanishing section $(\alpha,\beta) \in H^0(U, K_M^{1/2} \oplus K_M^{-1/2})$ and then

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \in H^0(U, E\otimes K_M).$$

So if L is φ-invariant, we must have that

$$\begin{pmatrix} 0 \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \lambda$$

for some section $\lambda \in H^0(U, K_M)$. Now suppose that there exists $p \in M$ such that $\alpha(p) \neq 0$. It follows, from the last equality, that $\lambda(p) = 0$ and that $\alpha(p) = 0$ which is a contradiction. Therefore α is the zero section and

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix} \otimes \lambda.$$

But since (α, β) is a non-vanishing section, β must be nowhere vanishing and λ should be, as well, the zero section. Thus, the only φ-invariant sub-bundle of (E, φ) can be $K_M^{-1/2}$.

If the transition functions for $K_M^{1/2}$ are given by $h_{ij} : U_{ij} \to \mathbb{C}^*$, then the transition functions for E and $\bigwedge^2 E$ are given by the matrix

$$\begin{pmatrix} h_{ij} & 0 \\ 0 & h_{ij}^{-1} \end{pmatrix}$$

and its determinant respectively. Thus $\bigwedge^2 E$ is the trivial line bundle over M, $\deg(E) = \deg(\bigwedge^2 E) = 0$ and

$$\mu(K_M^{-1/2}) = \deg(K_M^{-1/2}) = 1 - g < 0 = \mu(E)$$

meaning that E is stable.

By the third property of stability in Proposition 1.8, for a sufficiently small $\theta \in H^0(M, K_M^{\otimes 2})$, the Higgs bundle $(E, \tilde{\varphi})$ is stable with

$$\tilde{\varphi} = \begin{pmatrix} 0 & \theta \\ 1 & 0 \end{pmatrix}.$$

Consider the automorphism of E given by

$$\beta = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$$

for $\mu \in \mathbb{C}^*$. Then, by the first property in Proposition 1.8, for

$$\psi := \beta^{-1}\tilde{\varphi}\beta = \begin{pmatrix} 0 & \mu^{-1}\theta \\ \mu & 0 \end{pmatrix},$$

(E,ψ) is stable and by the second property of stability in Proposition 1.8 so is $(E,\mu^{-1}\psi)$. From this we conclude that the Higgs field, ϕ, from Example 1.4 is of the form $\mu^{-1}\psi$, that is,

$$\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mu^{-2}\theta \\ 1 & 0 \end{pmatrix} = \mu^{-1}\psi,$$

for some μ and θ. Therefore (E,ϕ) defines a point in the moduli space $\mathcal{M}(2,d)$.

Finally, for any $(E,\phi) \in \mathcal{M}(2,d)$ we have that $h(E,\phi) = (p_1(\phi) = \mathrm{Tr}(\phi), p_2(\phi) = \det(\phi))$. In particular, for the Higgs bundle from Example 1.4 we will have that

$$h(E,\phi) = (0,-\omega) \in H^0(M,K_M) \oplus H^0(M,K_M^{\otimes 2}) = \mathcal{A}(2,0).$$

Example 1.21. Consider the Higgs bundle from Example 1.5. To show that this Higgs bundle is stable can be done similarly as in the previous example. However we give ourselves the task of calculating the image of this Higgs bundle under the Hitchin fibration. By direct computation we have

$$\mathrm{Tr}(\phi) = \nu, \mathrm{Tr}(\phi^2) = \frac{\nu^{\otimes 2}}{3} + 2\mu \text{ and } \mathrm{Tr}(\phi^3) = \frac{\nu^{\otimes 3}}{9} + 2\nu \otimes \mu + 3\lambda.$$

Therefore, using the relations from Example 1.14, we can conclude that

$$h(E,\phi) = \left(p_1(\phi) = \nu, p_2(\phi) = \frac{\nu^{\otimes 2}}{3} - \mu, p_3(\phi) = \frac{\nu^{\otimes 3}}{27} - \frac{\nu \otimes \mu}{3} + \lambda \right) \in \mathcal{A}(3,0).$$

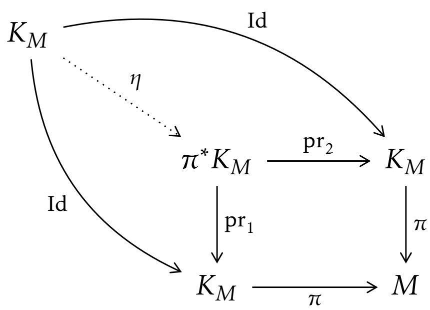
Chapter 2

Spectral description of Higgs bundles

In the first part of this chapter we will make sense of the characteristic polynomial of a Higgs bundle (E, ϕ) over M. This will lead our way to study spectral curves which broadly speaking correspond to the zero set of the characteristic polynomial. After studying some of its properties we will see the relation between the geometry of this curve and its corresponding Higgs bundle. Moreover, by means of the Jacobian variety of the spectral curve we will be able to understand and describe the fibers of the Hitchin fibration.

2.1 The characteristic polynomial of a Higgs bundle

Consider the commutative diagram



where $\mathrm{pr}_1, \mathrm{pr}_2 : \pi^* K_M \to K_M$ are the canonical projections and $\eta : K_M \to \pi^* K_M$ is the holomorphic section which is, in local coordinates, given by $(z, w) \mapsto w$ with z a local coordinate in M and $w \in \mathbb{C}$. So for a point $a = (a_1, \ldots, a_r) \in \mathcal{A}(r, d)$ we will have the section

$$\eta^{\otimes r} + \pi^* a_1 \otimes \eta^{\otimes r-1} + \ldots + \pi^* a_{r-1} \otimes \eta + \pi^* a_r \in H^0(K_M, \pi^* K_M^{\otimes r}) \tag{2.1}$$

that in the local coordinates (z, w) is given by the polynomial

$$w^r + a_1(z) w^{r-1} + \ldots + a_{r-1}(z) w + a_r(z). \tag{2.2}$$

Definition 2.1. Let (E, ϕ) be a Higgs bundle over M and $p_n(\phi) \in H^0(M, K^{\otimes n})$ as in Section 1.3. Then, if we set

$$a_n = (-1)^n p_n(\phi)$$

in Equation 2.1 we get a well defined global section in $H^0(K_M, \pi^* K_M^{\otimes r})$ which we call the *characteristic polynomial* of the Higgs bundle (E, ϕ) and denote by $\det(\mathrm{Id} \otimes \eta - \pi^* \phi)$.

Remark 2.2. $\det(\mathrm{Id} \otimes \eta - \pi^* \phi)$ only depends on the invariant polynomials $p_n(\phi)$, that is, on the image of $(E, \phi) \in \mathcal{M}(r, d)$ under the Hitchin fibration. Thus, for $(E_1, \phi_1), (E_2, \phi_2) \in \mathcal{M}(r, d)$, $\det(\mathrm{Id} \otimes \eta - \pi^* \phi_1) = \det(\mathrm{Id} \otimes \eta - \pi^* \phi_2)$ if and only if these two Higgs bundles are in the same fiber of the Hitchin Fibration. However, this does not mean that $(E_1, \phi_1) \cong (E_2, \phi_2)$.

Recall that the characteristic polynomial of $T|_W \in \mathrm{End}(W)$ divides the characteristic polynomial of $T \in \mathrm{End}(V)$ for V a finite dimensional complex vector space and $W \subset V$ a T-invariant vector sub-space. This is because if $\det(\lambda \mathrm{Id} - T) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_r)$, then for some $\{i_1, \ldots, i_m\} \subset \{1, \ldots, r\}$ we will have that $\det(\lambda \mathrm{Id} - T|_W) = (\lambda - \lambda_{i_1}) \ldots (\lambda - \lambda_{i_m})$. There is an analogous result in the Higgs bundles category as we are about to see in the following proposition.

Proposition 2.3. Let F be a ϕ-invariant sub-bundle of (E, ϕ). Then, the characteristic polynomial of $(F, \phi|_F)$ divides the characteristic polynomial of (E, ϕ).

Proof. Let

$$\det(\mathrm{Id}\otimes\eta-\pi^*\phi|_F)=\eta^{\otimes\tilde{r}}+\pi^*b_1\otimes\eta^{\otimes\tilde{r}-1}+\ldots+\pi^*b_{\tilde{r}}\in H^0(M,\pi^*K_M^{\otimes\tilde{r}}),$$

$$\det(\mathrm{Id}\otimes\eta-\pi^*\phi)=\eta^{\otimes r}+\pi^*a_1\otimes\eta^{\otimes r-1}+\ldots+\pi^*a_r\in H^0(M,\pi^*K_M^{\otimes r})$$

be the characteristic polynomials of $(F,\phi|_F)$ and (E,ϕ) respectively where $\tilde{r}=\mathrm{rank}(F)<\mathrm{rank}(E)=r$.

For all $n=1,\ldots,r-\tilde{r}$, consider the recursively defined quotients of sections:

$$c_n=\begin{cases} a_r/b_{\tilde{r}} & \text{if } n=r-\tilde{r},\\ \left(a_{r-k}-\sum\limits_{\substack{l+m=n\\ l\neq\tilde{r}}} b_l c_m\right)\Big/ b_{\tilde{r}} & \text{if } n=r-\tilde{r}-k,\ k=1,\ldots,r-\tilde{r}.\end{cases}$$

We are going to show that $c_n\in H^0(M,K_M^{\otimes n})$ for $n=1,\ldots,r-\tilde{r}$. Let $U_i,U_j\subseteq M$ be trivializing open sets for $K_M^{\otimes r-\tilde{r}}$ and $h_{ij}:U_{ij}\to\mathbb{C}^*$ the corresponding transition function. On the overlaps U_{ij} we have that

$$\left.\frac{a_r}{b_{\tilde{r}}}\right|_{U_i}=\frac{h_{ij}^{\otimes r}a_r|_{U_j}}{h_{ij}^{\otimes\tilde{r}}b_{\tilde{r}}|_{U_j}}=h_{ij}^{\otimes r-\tilde{r}}\left.\frac{a_r}{b_{\tilde{r}}}\right|_{U_j}$$

so that $c_{r-\tilde{r}}$ is a well defined meromorphic section of the line bundle $K_M^{\otimes r-\tilde{r}}$. In a similar fashion we can see, by induction, that c_n is a meromorphic section for $n=1,\ldots,r-\tilde{r}-1$. Now, we observe that these sections are indeed holomorphic or, equivalently, that these sections have no poles. This follows immediately from the discussion preceding this proposition since for any fixed $p\in M$, the polynomial $\det(\mathrm{Id}\otimes\eta-\pi^*\phi|_F)(p,\lambda)$, which can be regarded as the characteristic polynomial of the linear map $\phi|_{F,p}:F_p\to F_p\otimes K_{M,p}$, must divide the characteristic polynomial of $\phi_p:E_p\to E_p\otimes K_{M,p}$ which is $\det(\mathrm{Id}\otimes\eta-\pi^*\phi)(p,\lambda)$.

Finally, note that $\eta^{\otimes r-\tilde{r}}+\pi^*c_1\otimes\eta^{\otimes r-\tilde{r}-1}+\ldots+\pi^*c_{r-\tilde{r}}$ is a holomorphic section of the line bundle $K_M^{\otimes r-\tilde{r}}$ such that

$$\det(\mathrm{Id}\otimes\eta-\pi^*\phi|_F)\otimes(\eta^{\otimes r-\tilde{r}}+\pi^*c_1\otimes\eta^{\otimes r-\tilde{r}-1}+\ldots+\pi^*c_{r-\tilde{r}})=\det(\mathrm{Id}\otimes\eta-\pi^*\phi)$$

and this concludes the proof.

□

Hitchin's idea is to study the curve given by the zeroes of the characteristic polynomial that we constructed above. But, if one is to study the geometry of this curve, it is necessary to consider a compactified version of it.

2.2 Compactifying the spectral curve

In this section, we will think of M as a smooth, irreducible projective curve over $\mathbb{C}$. We start by compactifying the canonical bundle K_M to $\mathbb{P}(\mathcal{O}_M\oplus K_M)$. The latter is constructed by taking the projective space, $\mathbb{P}^1$, of the vector space corresponding to each fiber, $\mathbb{C}^2$, of the vector bundle bundle $\mathcal{O}_M\oplus K_M$. As a matter of fact, if we consider the natural projection

$$\tilde{\pi}:\mathbb{P}(\mathcal{O}_M\oplus K_M)\to M$$

for every point $p\in M$ there will exist an open set $U\subseteq M$ such that $\tilde{\pi}^{-1}(U)\cong U\times\mathbb{P}^1$. From a more algebro-geometric perspective, one can think of $\mathbb{P}(\mathcal{O}_M\oplus K_M)$ as the scheme $\mathbf{Proj}(\mathrm{Sym}(\mathcal{O}_M\oplus K_M^{-1}))$. For more details see Example A.12 and Theorem A.17.

Over $\mathbb{P}(\mathcal{O}_M\oplus K_M)$ consider the line bundle $\mathcal{O}(1)$ known as the *Serre twisting sheaf* or the *relatively ample bundle*. With respect to a local trivialization $U\times\mathbb{P}^1$ of $\mathbb{P}(\mathcal{O}_M\oplus K_M)$, this line bundle is given by the dual of the tautological bundle of $\mathbb{P}^1$, $\mathcal{O}(-1)$. One can show that $\tilde{\pi}_*\mathcal{O}(1)\cong\mathcal{O}_M\oplus K_M^{-1}$ [Har77, Chapter II, Proposition 7.11] which has a canonical section given by $p\mapsto(1,0)$. We denote by σ the image of this section under the isomorphism $H^0(M,\mathcal{O}_M\oplus K_M^{-1})\cong H^0(\mathbb{P}(\mathcal{O}_M\oplus K_M),\mathcal{O}(1))$. On the other hand, by the projection formula (see Remark A.5), we have that $\tilde{\pi}_*(\mathcal{O}(1)\otimes\tilde{\pi}^*K_M)\cong(\mathcal{O}_M\oplus K_M^{-1})\otimes K_M\cong K_M\oplus\mathcal{O}_M$. This bundle has as well a canonical section given by $p \mapsto (0,1)$ and its image under the isomorphism $H^0(M, K \oplus \mathcal{O}_M) \cong H^0(\mathbb{P}(\mathcal{O}_M \oplus K_M), \mathcal{O}(1) \otimes \tilde{\pi}^* K_M)$ will be denoted by η.

For a point $a = (a_1, \dots, a_r)$ in the Hitchin base $\mathcal{A}(r,d)$, consider the section

$$\eta^{\otimes r} + \tilde{\pi}^* a_1 \otimes \eta^{\otimes r-1} \otimes \sigma + \ldots + \tilde{\pi}^* a_r \otimes \sigma^{\otimes r} \in H^0(\mathbb{P}(\mathcal{O}_M \oplus K_M), \mathcal{O}(r) \otimes \tilde{\pi}^* K_M^{\otimes r}). \tag{2.3}$$

Intuitively, we can think of this section as the homogenization of a polynomial in the affine case.

Definition 2.4. Let X_a be the zero set of the section in Equation 2.3. This set is called the *spectral curve* associated to $a \in \mathcal{A}(r,d)$.

Remark 2.5. Regarding the above construction, in [BNR89] and [Hit87b] it is proved that:

- There is an open dense subset in the Hitchin base $\mathcal{A}(r,d)$ for which X_a is a smooth projective curve. In other words, for a generic point a in the Hitchin base, X_a can be given structure of compact Riemann surface.
- When restricted to $K_M \hookrightarrow \mathbb{P}(K_M \oplus \mathcal{O}_M)$, η is the canonical section in Equation 2.1.
- The sections η and σ are disjoint and σ restricted to X_a is a non-vanishing section of $\mathcal{O}(1)|_{X_a}$ so that $\mathcal{O}(1)$ restricted to the spectral curve X_a is the trivial bundle. Thus, X_a can be seen as the zero set of the section in Equation 2.1.

Actually, we can be more precise in describing the set of points in the Hitchin base for which the spectral curve corresponding to a Higgs bundle is smooth or, equivalently, a compact Riemann surface.

Proposition 2.6. For $h(E,\phi) = (a_1, \dots, a_r) \in \mathcal{A}(r,d)$, the corresponding spectral curve X_a is smooth if and only if at every point which is a multiple zero of a_r, the section a_{r-1} is nonzero [BNR89, Remark 3.5].

Proof. The key fact is that the spectral curve, which is given locally by the zero set of Equation 2.2, is smooth at a point if and only if the Jacobian at this point is non-zero

[Liu02, Chapter 4, Theorem 2.19].

First, let us assume that the spectral curve corresponding to the Higgs pair (E,ϕ) is smooth. We can think of the points on the spectral curve $X_a \subset K_M$ as pairs (p,λ) where $p \in M$ and λ is an eigenvalue of the linear map $\phi_p : E_p \to E_p \otimes K_{M,p}$. So if $(p,\lambda) \in X_a$ is such that p is a multiple zero of a_r, then ϕ_p has vanishing determinant and zero is an eigenvalue of ϕ_p. Therefore, $(p,0) \in X_a$ and the Jacobian matrix at this point will be given by

$$J_{X_a}(p,0) = \begin{pmatrix} a_1'(z)w^{r-1} + \ldots + a_r'(z) & rw^{r-1} + \ldots + a_{r-1}(z) \end{pmatrix}\Big|_{(p,0)} = \begin{pmatrix} 0 & a_{r-1}(p) \end{pmatrix}.$$

But, since the spectral curve is smooth, the Jacobian must be non-zero which means that $a_{r-1}(p) \neq 0$ as desired.

Now we go in the other direction. Assume that X_a is not smooth, then the Jacobian matrix is degenerate for some points in the spectral curve. At each one of these we choose coordinates centered at the origin so that the degeneracy of the Jacobian gives

$$J_{X_a}(p,0) = \begin{pmatrix} a_r'(p) & a_{r-1}(p) \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Thus, $a_r(p) = a_r'(p) = a_{r-1}(p) = 0$ concluding the proof.

□

Definition 2.7. We denote the restriction of $\pi : K_M \to M$ to the spectral curve X_a as $\pi_a : X_a \to M$ and we call it *spectral cover*.

Remark 2.8. For any $p \in M$, the fiber $\pi_a^{-1}(p)$ is, in local coordinates, the zero set of the polynomial in Equation 2.2 for a fixed z. This polynomial has r roots counted with multiplicity which means that for a generic point $a \in \mathcal{A}(r,d)$, the spectral cover $\pi_a : X_a \to M$ is an r-sheeted covering of compact Riemann surfaces. Or, equivalently, that the degree of the spectral cover is r.

2.3 Another point of view on spectral curves

An alternative description of the spectral curve, which will be helpful to calculate its genus, is given as follows. For a point $a = (a_1, \dots, a_r) \in \mathcal{A}(r,d)$, consider the $\mathcal{O}_M$-module homomorphism

$$\begin{aligned} f : K_M^{\otimes -r} &\to \mathrm{Sym}(K_M^{-1}) = \bigoplus_{n=0}^{\infty} K_M^{\otimes -n} \\ \tilde{\eta} &\mapsto \sum_{i=1}^{r} a_i \otimes \tilde{\eta} \end{aligned}$$

with $a_0 = 1 \in H^0(M, \mathcal{O}_M)$. If $\mathcal{I}_a \subset \mathrm{Sym}(K_M^{-1})$ is the ideal sheaf given by the image of f, then

$$\mathrm{Sym}(K_M^{-1})/\mathcal{I}_a \cong \bigoplus_{n=0}^{r-1} K_M^{\otimes -n}.$$

It can be shown that $X_a = \mathbf{Spec}(\mathrm{Sym}(K_M^{-1})/\mathcal{I}_a) \subset \mathbf{Spec}(\mathrm{Sym}(K_M^{-1})) = K_M$ [BNR89]. Indeed, let us give a local picture of this construction. Let $U \subseteq M$ be a trivializing open set of K_M^{-1} and $\tilde{\eta} \in H^0(U, K_M^{-1})$ a nowhere vanishing holomorphic section. As we saw in Example A.12, we have that

$$K_M|_U \cong \mathrm{Spec}(\mathcal{O}_M(U)[\tilde{\eta}]).$$

Therefore,

$$X_a|_U \cong \mathrm{Spec}(\mathcal{O}_M(U)[\tilde{\eta}]/(\tilde{\eta}^{\otimes r} + \tilde{a}_1 \otimes \tilde{\eta}^{\otimes r-1} + \dots + \tilde{a}_n))$$

with $\tilde{a}_n = \tilde{\eta}^{\otimes n}(a_n) \in \mathcal{O}_M(U)$ for $n = 1, \dots, r$.

In the following proposition we will compute, as promised, the genus of the spectral curve.

Proposition 2.9. The genus of the spectral curve X_a is given by

$$g_a = r^2(g-1) + 1.$$

Proof. We know that

$$\chi(X_a, \mathcal{O}_{X_a}) = 1 - g_a.$$

By Remark A.4, Euler characteristic is preserved under direct images and, by Theorem A.11, $\pi_{a*}(\mathcal{O}_{X_a}) \cong \bigoplus_{n=0}^{r-1} K_M^{\otimes -n}$ so we have

$$\begin{aligned} g_a &= 1 - (\dim(H^0(M, \pi_{a*}\mathcal{O}_{X_a})) - \dim(H^1(M, \pi_{a*}\mathcal{O}_{X_a}))) \\ &= 1 - \sum_{n=0}^{r-1} \dim(H^0(M, K_M^{\otimes -n})) - \dim(H^1(M, K_M^{\otimes -n})). \end{aligned}$$

Riemann-Roch theorem, for all $n = 0, \dots, r-1$, yields

$$\dim(H^0(M, K_M^{\otimes -n})) - \dim(H^1(M, K_M^{\otimes -n})) = -n(2g-2) + (1-g) = (2n+1)(1-g).$$

Hence,

$$g_a = 1 - \sum_{n=0}^{r-1} (2n+1)(1-g) = 1 + r^2(g-1).$$

□

2.4 The Beauville-Narasimhan-Ramanan correspondence

The main goal of this section is to characterize the generic fibers of the Hitchin fibration. More precisely, we will see that for a point $a \in \mathcal{A}(r,d)$, there is a bijection between the Higgs bundles in $h^{-1}(a) \subset \mathcal{M}(r,d)$ and the Jacobian variety of the spectral curve X_a, which is a g_a-dimensional complex torus. But first, let us discuss the relation between the geometry of the spectral cover $\pi_a : X_a \to M$ associated to a Higgs pair (E, ϕ) and the eigenvalues of the family of linear maps $\phi_p : E_p \to E_p \otimes K_{M,p}$ with $p \in M$.

Let us denote the restriction of the canonical section $\eta \in H^0(K_M, \pi^* K_M)$ to X_a with the same symbol. Consider the section

$$D := r\eta^{\otimes r-1} + (r-1)\pi_a^* a_1 \otimes \eta^{\otimes r-2} + \ldots + \pi_a^* a_{r-1} \in H^0(K_M, \pi_a^* K_M^{\otimes r-1})$$

which can be interpreted, fiberwise, as the derivative of the characteristic polynomial of the linear map $\phi_p : E_p \to E_p \otimes K_{M,p}$. Recall that the points in the spectral curve X_a can be understood as pairs (p,λ) where $\lambda \in K_{M,p}$ is an eigenvalue of the linear map $\phi_p : E_p \to E_p \otimes K_{M,p}$. So $Q := (p,\lambda) \in X_a$ corresponds to a multiple eigenvalue of ϕ_p if and only if D vanishes in Q. As a matter of fact, the multiplicity of Q as a zero of D is deeply related with the multiplicity or branching order [Mir95, Chapter II, Definition 4.2] of the spectral cover at this point, $\mathrm{mult}_Q(\pi_a)$.

Theorem 2.10. The section $D \in H^0(X_a, \pi_a^* K_M^{\otimes r-1})$ vanishes at $Q \in X_a$ with multiplicity $\mathrm{mult}_Q(\pi_a) - 1$.

Proof. The proof is divided in two parts. First, we will see that $Q \in X_a$ is a branching point of the spectral cover $\pi_a : X_a \to M$ if and only if D vanishes at Q. Then, we will show that if $Q \in X_a$ is a zero of D, then it is a zero of order $\mathrm{mult}_Q(\pi_a) - 1$.

As we saw in Section 2.1, with respect to local coordinates (z,w) the spectral curve X_a can be described as the zero set of Equation 2.2, that is, the set pairs (z,w) such that

$$f(z,w) := w^r + a_1(z)w^{r-1} + \ldots + a_{r-1}(z)w + a_r(z) = 0.$$

Note that the derivative

$$\frac{\partial f}{\partial w} = rw^{r-1} + (r-1)a_1(z)w^{r-2} + \ldots + a_{r-1}(z)$$

is precisely the section D written in the local coordinates we have chosen. Hence, the zeroes of the section D correspond to points $Q := (z_0, w_0) \in \mathbb{C}^2$ such that $(\partial f/\partial w)(Q) = 0$.

If $(\partial f/\partial w)(Q) \neq 0$, by the Implicit Function theorem, there exists an open set $U \subseteq \mathbb{C}$ containing z_0 in which $w = g(z)$. Thus $\pi_a(z,w) = z$ is a chart for X_a around Q. Since every chart is a biholomorphism we will have that $\mathrm{mult}_Q(\pi_a) = 1$ or, equivalently, that π_a does not branch at this point. Conversely, if $(\partial f/\partial w)(Q) = 0$ we must have that $(\partial f/\partial z)(Q) \neq 0$ given that X_a is smooth. Again, by the Implicit Function theorem, in an open set $U \subseteq \mathbb{C}$ containing w_0, X_a is locally the graph of a holomorphic function $g(w)$. Moreover, $f(g(w),w) = 0$

and

$$\frac{\partial f}{\partial w} = -\frac{\partial f}{\partial z} g'(w)$$

in U. Therefore

$$\frac{\partial f}{\partial z}(Q) g'(w_0) = 0$$

which implies that $g'(w_0) = 0$. But notice that in U, $\pi_a(g(w), w) = g(w)$ so π_a ramifies at Q concluding the proof of the first part.

To see that the multiplicity of each zero of the section D is $\text{mult}_Q(\pi_a) - 1$ is straigthforward from what we just did. In a branching point of π_a or zero of D, $Q \in X_a$, $g(w)$ is the local expression of the spectral cover π_a as we just saw. Hence, the order of a zero of $g(w)$ is the multiplicity or branching order of that zero with respect to π_a. Since $(\partial f/\partial z)(Q) \neq 0$, $(\partial f/\partial w)(Q)$ vanishes if and only if $g'(w_0)$ does and they do both with the same multiplicity. Hence $\partial f/\partial w$ vanishes at Q with multiplicity $\text{mult}_Q(\pi_a) - 1$.

□

Remark 2.11. The previous theorem can be interpreted as follows. If the spectral cover $\pi_a : X_a \to M$ branches at a point $(p, \lambda) \in X_a$, then λ is a multiple eigenvalue of the linear map $\phi_p : E_p \to E_p \otimes K_{M,p}$ with multiplicity being the branching order, $\text{mult}_Q(\pi_a)$.

Other interesting facts follow from Theorem 2.10. These have to do with the ramification divisor of the spectral cover $\pi_a : X_a \to M$ and the canonical bundle of the spectral curve, K_{X_a}. Recall that the ramification divisor, Δ, of the spectral cover $\pi_a : X_a \to M$ is given by

$$\Delta = \sum_{Q \in X_a} (\text{mult}_Q(\pi_a) - 1) \cdot Q \tag{2.4}$$

with $\text{mult}_Q(\pi_a)$ the multiplicity or branching order of the spectral cover at the point $Q \in X_a$ [Mir95, Chapter V, Definition 1.18]. This sum must be finite since both X_a and M are compact.

In the following proposition we will see that the line bundle given by Δ can be understood in terms of a well known line bundle over the Riemann surface M.

Corollary 2.12. Let $\mathcal{O}(\Delta)$ be the line bundle over X_a associated to the ramification divisor Δ of Equation 2.4, then

$$\mathcal{O}(\Delta) \cong \pi_a^* K_M^{\otimes r-1}.$$

Proof. From Theorem 2.10, observe that $\mathrm{div}(D) = \Delta$. This means that these two divisors are linearly equivalent and that their corresponding line bundles, $\pi_a^* K_M^{\otimes r-1}$ and $\mathcal{O}(\Delta)$ respectively, are isomorphic. □

Corollary 2.13. Let K_{X_a} be the canonical bundle of the spectral curve X_a, then

$$K_{X_a} \cong \pi_a^* K_M^{\otimes r}.$$

Proof. Hurwitz's theorem [Har77, Chapter IV, Proposition 2.3] relates the canonical divisors of X_a and M via the ramification divisor Δ. In particular, this theorem gives

$$K_{X_a} \cong \pi_a^* K_M \otimes \mathcal{O}(\Delta)$$

so that the desired isomorphism follows from the previous corollary. □

Remark 2.14. If the spectral cover does not have branching points, $\mathcal{O}(\Delta)$ is then the trivial line bundle over X_a. This implies, by Corollary 2.12, that $\pi_a^* K_M^{\otimes r-1} \cong \mathcal{O}_{X_a}$. But then

$$r(r-1)(2g-2) = \deg(\pi_a^* K_M^{\otimes r-1}) = \deg(\mathcal{O}_{X_a}) = 0$$

and, since $g > 1$, we must have that $r = 1$. This is not surprising since for all $p \in M$, the linear map $\phi_p : E_p \to E_p \otimes K_{M,p}$, when (E, ϕ) is a rank 1 Higgs bundle (see Example 1.1), trivially has only one eigenvalue so there are none multiplicities to be counted. Far more interesting is that this also implies that for a Higgs bundle of rank $r \geq 2$, the corresponding spectral cover will always have branching points.

Remark 2.15. The preceding corollaries give us another way to work out the genus of the spectral curve X_a. By the Riemann-Roch theorem we have

$$g_a - 1 = \dim(H^0(X_a, K_{X_a})) - \dim(H^1(X_a, K_{X_a})) = \deg(K_{X_a}) + (1 - g_a)$$
$$= r^2(2g-2) + (1 - g_a)$$

and then $g_a = r^2(g-1)+1$ which is the value obtained in Proposition 2.9.

Theorem 2.16. Let $a = (a_1, \ldots, a_r) \in \mathcal{A}(r,d)$ such that the corresponding spectral curve, X_a is a Riemann surface. Then there is a bijective correspondence between line bundles $L \in \mathrm{Pic}_0(X_a)$ and isomorphism classes of Higgs bundles $(E, \phi) \in \mathcal{M}(r,d)$ with characteristic polynomial, $\det(\mathrm{Id} \otimes \eta - \pi^*\phi)$, having coefficients $a_1, \ldots, a_r$ [BNR89, Proposition 3.6].

Remark 2.17. The line bundles in this bijection correspond to everywhere regular Higgs pairs. The regularity condition means that for all $p \in M$, $\phi_p : E_p \to E_p \otimes K_{M,p}$ is a linear map whose Jordan normal form contains a single Jordan block for each eigenvalue or, in other words, that each eigenspace of the linear map ϕ_p is one-dimensional [Bea90, Section 1.12].

Proof. Let $a = (a_1, \ldots, a_r) \in \mathcal{A}(r,d)$ such that the corresponding spectral curve X_a given by $\eta^{\otimes r} + \pi^* a_1 \otimes \eta^{\otimes r-1} + \ldots + \pi^* a_{r-1} \otimes \eta + \pi^* a_r = 0 \subset K_M$ is a Riemann surface and let $\pi_a : X_a \to M$ the r-sheeted spectral cover.

Let L be a line bundle over the spectral curve X_a such that $\deg(L) = d + (r - r^2)(1-g)$. We will construct from L a Higgs bundle in the moduli space $\mathcal{M}(r,d)$. Given that $\deg(\pi_a) = r$, the direct image $\pi_{a*}L$ will be an holomorphic vector bundle of rank r. Recall, by Definition A.1, that $H^0(U, \pi_{a*}L) = H^0(\pi_a^{-1}(U), L)$ and, by the projection formula in Proposition A.2, that $H^0(U, \pi_{a*}L \otimes K_M) = H^0(\pi_a^{-1}(U), L \otimes \pi_a^* K_M)$ for all open set $U \subseteq M$. On the other hand, tensorization by the canonical section $\eta \in H^0(M, \pi_a^* K_M)$ induces a homomorphism

$$H^0(\pi_a^{-1}(U), L) \xrightarrow{\eta} H^0(\pi_a^{-1}(U), L \otimes \pi_a^* K_M)$$

which, in view of the previous identifications, can be pushed down to define a Higgs field

$$\pi_{a*}\eta : \pi_{a*}L \to \pi_{a*}L \otimes K_M.$$

Thus, the pair $(\pi_{a*}L, \pi_{a*}\eta)$ is a Higgs bundle which, by construction, satisfies the equation defining X_a and whose degree is, by the formula in Proposition A.3,

$$\begin{aligned}\deg(\pi_{a*}L) &= \deg(L) + (1 - g_a) - \deg(\pi_a)(1-g) \\ &= d + (r - r^2)(1-g) + 1 - (1 + r^2(g-1)) - r(1-g) \\ &= d.\end{aligned}$$

It remains to show that the Higgs bundle $(\pi_{a*}L, \pi_{a*}\eta)$ is stable. By Proposition 2.3, the characteristic polynomial of any invariant sub-bundle would divide the characteristic polynomial of the Higgs field $\pi_{a*}\eta$. But the spectral curve X_a is smooth and therefore irreducible, so there are no $\pi_{a*}\eta$-invariant sub-bundles of $(\pi_{a*}L, \pi_{a*}\eta)$ [Sch13, Chapter 2, Section 2.1.2]. Hence, the Higgs bundle $(\pi_{a*}L, \pi_{a*}\eta)$ defines a point in $\mathcal{M}(r,d)$.

Now, we go in the other direction. Let $(E,\phi) \in \mathcal{M}(r,d)$ with spectral curve given by X_a. As we have been discussing, the points of the spectral curve are given by pairs (p,λ) where $p \in M$ and $\lambda \in K_{M,p}$ is an eigenvalue of the linear map $\phi_p : E_p \to E_p \otimes K_{M,p}$. By attaching to each point in the spectral curve its corresponding eigenline space we obtain a line sub-bundle $L' \hookrightarrow \pi_a^* E$ over the spectral curve X_a. So the line bundle associated to (E,ϕ) is then $L = L' \otimes \mathcal{O}(\Delta) \cong L' \otimes \pi_a^* K_M^{\otimes r-1}$.

The two constructions that we have just discussed are inverse to one another. Moreover, by tensoring the line bundles L with a fixed line bundle over X_a with the proper degree, we obtain points in the Jacobian variety of X_a, $\mathrm{Pic}_0(X_a)$, and this concludes the proof.

□

Remark 2.18. Let L a line bundle over the spectral curve X_a as in the previous theorem and $(E,\phi) := (\pi_{a*}L, \pi_{a*}\eta)$ the Higgs bundle obtained by taking direct images. Then, the above correspondence can be seen by means of the exact sequence [BNR89, Remark 3.7]:

$$0 \longrightarrow L\otimes \pi_a^* K_M^{\otimes 1-r} \longrightarrow \pi_a^* E \xrightarrow{\pi_a^*\phi-\eta} \pi_a^*(E\otimes K_M) \longrightarrow L\otimes \pi_a^* K_M \longrightarrow 0.$$

Example 2.19. Let $a=(0,a_2)\in \mathcal{A}(2,d)$. The corresponding spectral curve, X_a, is given by

$$\eta^{\otimes 2} + \pi^* a_2 = 0 \subset K_M.$$

For a generic choice of $a_2 \in H^0(M, K_M^{\otimes 2})$, X_a is a compact Riemann surface and, by Remark 2.8, the associated spectral cover, $\pi_a = X_a \to M$, will be a 2-sheeted branched covering.

Let L be a holomorphic line bundle over the spectral curve X_a. For every ramification point $q\in \pi_a^{-1}(p)$ one can find open sets U, V, which contain p and q respectively, such that the map $\pi_a|_V : V \to U$ is, in local coordinates, given by $z \mapsto z^2 := w$. A section of L over V can be seen, in local coordinates, as a holomorphic function

$$h(z) = \sum_{n=0}^{\infty} a_n z^n = h_0(w) + z h_1(w).$$

From the previous theorem, the Higgs bundle associated to L is given by the pair $(E,\phi) := (\pi_{a*}L, \pi_{a*}\eta)$. In particular, recall that the Higgs field was obtained via tensorization by the canonical section η which, in the local coordinates we have chosen, is given by $z\mapsto z$. Therefore,

$$\phi(h_0(w) + z h_1(w)) = z h_0(w) + w h_1(w).$$

and, with respect to the local frame $\{1, z\}$, the Higgs field will be given by

$$\phi = \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}.$$

For instance, if $L = \pi_a^* K_M^{1/2}$, then by the projection formula in Proposition A.2 and Theorem A.11 we have that

$$\pi_{a*}L \cong \pi_{a*}\mathcal{O}_{X_a} \otimes K_M^{1/2} \cong (\mathcal{O}_M \oplus K_M^{-1}) \otimes K_M^{1/2} \cong K_M^{1/2} \oplus K_M^{-1/2}$$

which corresponds to the Higgs bundle studied in Example 1.4.

Appendix A

Since the purpose of this work is to be the most self-contained possible, we consider appropriate to briefly introduce many concepts that play an important role in the previous constructions and discussions. More precisely, on those regarding the spectral curve introduced in Chapter 2. Mostly, no proofs are given but we give precise references for the interested reader to follow.

A.1 Direct images of line bundles

Let $f : X \to Y$ be a holomorphic map between compact Riemann surfaces. We will study, in the first part of this appendix, how to produce vector bundles on Y out of line bundles over X.

Definition A.1. Let $\mathcal{G}$ be a sheaf on X, we define the *direct image sheaf* $f_*\mathcal{G}$ on Y by

$$(f_*\mathcal{G})(U) = \mathcal{G}(f^{-1}(U))$$

for all open set $U \subseteq Y$.

Proposition A.2. Let $\mathcal{O}_X(L)$ be the sheaf of holomorphic sections of a line bundle L over X, then [Hit99, Proposition 4.2]:

1. $H^0(Y, f_*\mathcal{O}_X(L)) \cong H^0(X, \mathcal{O}_X(L))$,

2. There exist a holomorphic vector bundle E over X, of rank $r := \deg(f)$, such that $f_*\mathcal{O}_X(L) = \mathcal{O}_Y(E)$. Or, in other words, $f_*\mathcal{O}_X(L)$ is a locally free sheaf of rank r.

3. *(Projection formula)* If V is a holomorphic vector bundle on Y, then

$$f_*\mathcal{O}_X(L \otimes f^*V) \cong \mathcal{O}_Y(E \otimes V)$$

with f^*V the pullback bundle of V through f.

Proof. The first part is straightforward setting $U = Y$ because then, by definition, $f_*\mathcal{O}_X(L)(Y) = \mathcal{O}_X(L)(f^{-1}(Y)) = \mathcal{O}_X(L)(X)$. For the second part, it is enough to show that for every $p \in Y$, there exists a neighbourhood U for which we have an isomorphism

$$f_*\mathcal{O}_X(L)(U) \cong \underbrace{\mathcal{O}_Y(U) \oplus \ldots \oplus \mathcal{O}_Y(U)}_{r \text{ times}}.$$

If p is a regular value the result is straightforward. Note that $f^{-1}(p)$ consist of r distinct points and since $f' \neq 0$ at all of these, there are r disjoint open sets $U_i \subset X$ which are biholomorphic to $U \subseteq Y$ through f. Thus, in this case

$$f_*\mathcal{O}_X(L)(U) = \mathcal{O}_X(L)(f^{-1}(U)) = \bigoplus_{j=1}^{r} \mathcal{O}_X(U_j).$$

In the general case, $f^{-1}(p)$ contain branch points. For every $q \in f^{-1}(p)$ one can find open sets $\tilde{U}, U$, which contain $f^{-1}(p)$ and p respectively, such that the map $f : \tilde{U} \to U$ is, in local coordinates, given by $z \mapsto w := z^k$. A section of L over $\tilde{U}$, in local coordinates, will look like

$$h(z) = \sum_{n=0}^{\infty} a_n z^n = h_0(w) + z h_1(w) + \ldots + z^{k-1} h_{k-1}(w),$$

so the space of sections is the direct sum of k copies of holomorphic functions on U. The total multiplicity of the branch points still satisfies

$$\sum_{q \in f^{-1}(p)} k_q = \deg(f) = r$$

so in this case too, local sections of $f_*\mathcal{O}(L)$ look like r local holomorphic functions. It follows that $f_*\mathcal{O}(L)$ is the sheaf of sections of a rank r holomorphic vector bundle, E, over Y.

For the last part of the proposition we follow the lines of Gunning [Gun67, Lemma 10]. Assume that V is a rank r vector bundle so with respect to a trivializing open set $U \subseteq Y$, it can be assumed that $\mathcal{O}_Y(V) \cong \bigoplus_{n=1}^{r} \mathcal{O}_Y$ with $\mathcal{O}_Y$ the sheaf of holomorphic functions of the Riemann surface Y. Then, $\mathcal{O}_Y(E \otimes V) = \mathcal{O}_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(V) = \bigoplus_{n=1}^{r} \mathcal{O}_Y(E)$ and, on the other hand, $f^*(\mathcal{O}_Y(V)) = f^*(\bigoplus_{n=1}^{r} \mathcal{O}_Y) = \bigoplus_{n=1}^{r} \mathcal{O}_X$ so that $\mathcal{O}_X(L) \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y(V) = \bigoplus_{n=1}^{r} \mathcal{O}_X(L)$. Thus,

$$f_*(\mathcal{O}_X(L) \otimes f^*\mathcal{O}_Y(V)) = f_*\left(\bigoplus_{n=1}^{r} \mathcal{O}_X(L)\right) = \bigoplus_{n=1}^{r} \mathcal{O}_Y(E) = \mathcal{O}_Y(E \otimes V)$$

since direct image commutes with direct sums [Gun67, Lemma9]. □

One can work out one of the most important topological invariants of E, its degree, as we will see in the following proposition.

Proposition A.3. With L and E as in the last proposition,

$$\deg E = \deg(L) + (1 - g_X) - \deg(f)(1 - g_Y)$$

for g_X and g_Y the genus of X and Y respectively [Hit99, Proposition 4.3].

Remark A.4. We can rewrite the result of the last proposition in terms of the Euler characteristic. By rearranging the terms and applying the Riemann-Roch theorem we have

$$\chi(X, \mathcal{O}(L)) = \deg(L) + (1 - g_X) = \deg(E) + \operatorname{rank}(E)(1 - g_Y) = \chi(Y, \mathcal{O}(E)).$$

This means that the Euler characteristic is preserved under direct images.

Remark A.5. One can define define direct images in a similar way for ringed spaces $(X, \mathcal{O}_X)$ and algebraic schemes and in this setting the properties that we have just discussed also hold. See for instance [Har77, Chapter II, Section 5].

A.2 Sheaves of modules

Now, we review some facts about sheaves of modules that are useful in order to work with spectral curves. The main reference here is Hartshorne's book [Har77, Chapter II, Section 3, Section 5].

Definition A.6. A morphism of schemes $\pi : E \to M$ is *affine* if there exists an open affine cover $\{U_i\}$ of M such that $\pi^{-1}(U_i)$ is affine for every i.

Definition A.7. A morphism between schemes $\pi : E \to M$ is *finite* if there exists a covering of M by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, $\pi^{-1}(V_i)$ is affine, equal to $\operatorname{Spec} A_i$ where A_i is a B_i-algebra which is a finitely generated B_i-module.

Remark A.8. It is straighforward to check from the definitions that every finite morphism is affine.

Definition A.9. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_M$-modules. The *symmetric algebra* of $\mathcal{F}$ is given by the sheafification of the presheaf which to each open set $U \subseteq M$ assigns $\operatorname{Sym}(\mathcal{F}(U))$ as an $\mathcal{O}_M(U)$-module. The result of this process is what we call an $\mathcal{O}_M$-algebra whose component in each degree is an $\mathcal{O}_M$-module and we denote it by $\operatorname{Sym} \mathcal{F}$

Theorem A.10. Let M be a scheme and $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_M$-algebras. There exist a unique scheme $\mathbf{Spec}\ \mathcal{F}$ and a unique morphism $\pi : \mathbf{Spec}\ \mathcal{F} \to M$ such that for every affine open set $V \subseteq M$, $\pi^{-1}(V) \cong \operatorname{Spec} \mathcal{F}(V)$ and for every inclusion $U \hookrightarrow V$ of open affines of M, the morphism $\pi^{-1}(U) \hookrightarrow \pi^{-1}(V)$ corresponds to the restriction morphism $\mathcal{F}(V) \to \mathcal{F}(U)$. This scheme is called the *relative Spec* of the sheaf $\mathcal{F}$.

Theorem A.11. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_M$-algebra, then $\pi : E := \mathbf{Spec}\ \mathcal{F} \to M$ is an affine morphism and $\mathcal{F} \cong \pi_* \mathcal{O}_E$. Conversely, if $\pi : E \to M$ is an affine morphism, then $\mathcal{F} := \pi_*(\mathcal{O}_E)$ is a quasi-coherent sheaf of $\mathcal{O}_M$-algebras and $E \cong \mathbf{Spec}\ \mathcal{F}$.

Example A.12. Let $\pi : E \to M$ be a rank r vector bundle over a complex scheme M. Consider the dual vector bundle E^* and let $\mathcal{O}(E^*)$ be its sheaf of sections. We will see that E can be identified with $\mathbf{Spec}(\operatorname{Sym} \mathcal{O}(E^*))$.

We will first show that $\rho : \mathbf{Spec}\ \mathrm{Sym}(\mathcal{O}(E^*)) \to M$ defines a rank r vector bundle over M. We know that $\mathcal{O}(E^*)$ is a locally free $\mathcal{O}_M$-module of rank r. For each open subset $U \subseteq M$ for which $\mathcal{O}(E^*)$ is free, choose a basis, $\{x_1, \ldots, x_r\}$, of $\mathcal{O}(E^*)|_U$. Let $\varphi : \rho^{-1}(U) \to U \times \mathbb{A}^r$ be the isomorphism resulting from the identification of $\mathrm{Sym}(\mathcal{O}(E^*)(U))$ with $\mathcal{O}_M(U)[x_1, \ldots, x_r]$. Thus, $\pi : \mathbf{Spec}\ \mathrm{Sym}(\mathcal{O}(E^*)) \to M$ with the given trivializations defines a line bundle over M.

Now, we will show that $E \cong \mathbf{Spec}\ \mathrm{Sym}(\mathcal{O}(E^*))$. To see this we will check that the sheaf of sections of both vector bundles are isomorphic. Every $\sigma \in \mathcal{O}(E)(U)$ can be regarded as an element in $\mathrm{Hom}(\mathcal{O}(E^*)|_U, \mathcal{O}_U)$ by considering the map $\tau \mapsto \tau(\sigma)$. Thus, σ induces an algebra homomorphism $\mathrm{Sym}(\mathcal{O}(E^*)|_U) \to \mathcal{O}_U$ which gives us a morphism $\tilde{\sigma} : U = \mathbf{Spec}(\mathcal{O}_U) \to \mathbf{Spec}\ \mathrm{Sym}(\mathcal{O}(E^*)|_U) = \rho^{-1}(U)$ and this is a section of the vector bundle $\mathbf{Spec}\ \mathrm{Sym}\ (\mathcal{O}(E^*))$. The desired isomorphism is then given by $\sigma \mapsto \tilde{\sigma}$.

A.3 The Proj construction

The last part of this appendix is devoted to the **Proj** construction. This is useful to make sense of what the projectivization of a vector bundle means from an algebro-geometric perspective. The main reference that we follow here is Qing Liu's book [Liu02, Section 2.3, Section 8.1].

Let R be a ring and $B = \bigoplus_{d \geq 0} B_d$ a graded R-algebra. An ideal I of B is said to be *homogeneous* if it is generated by homogeneous elements and this is equivalent to say that $I = \bigoplus_{d \geq 0} (I \cap B_d)$. The set of homogeneous prime ideals of B, that do not contain the ideal $B_+ := \bigoplus_{d > 0} B_d$, is denoted by $\mathrm{Proj}(B)$. For any homogeneous ideal I of B, we let $V_+(I)$ denote the set of prime ideals $\mathfrak{p} \in \mathrm{Proj}(B)$ containing I. So, we have the equalities

$$\bigcap_\alpha V_+(I_\alpha) = V_+\left(\sum_\alpha I_\alpha\right), \ V_+(I) \cup V_+(J) = V_+(I \cap J), V_+(B) = \varnothing, \ V_+(0) = \mathrm{Proj}(B)$$

which make possible to endow Proj(B) with a topology whose closed sets are of the form $V_+(I)$. This topology is the *Zariski topology* on Proj(B).

For every homogeneous element $f \in B$, we define $D_+(f) = \mathrm{Proj}(B) \setminus V_+(fB)$. These open sets are called *principal open sets* and they constitute an open base for the Zariski topology on Proj(B). In fact, one can restrict to the open sets of the form $D_+(f)$ with $f \in B_+$. The following theorem tells us about the scheme structure of Proj(B).

Theorem A.13. Let R be a ring and B a graded R-algebra. Then we can endow Proj(B) with a unique structure of R-scheme such that for any homogeneous element $f \in B_+$, the open set $D_+(f)$ is affine and isomorphic to Spec $B_{(f)}$. $B_{(f)}$ stands for the subring of B_f whose elements are of the form gf^{-n} with $n \geq 0$ and $\deg(g) = n\deg(f)$ [Liu02, Chapter 2, Proposition 3.38].

Example A.14. Let $U_1 = \mathrm{Spec}(\mathbb{C}[x_1^{-1}x_2]) \cong \mathbb{A}^1$ and $U_2 = \mathrm{Spec}(\mathbb{C}[x_2^{-1}x_1]) \cong \mathbb{A}^1$. For the open subschemes $U_{ij} = U_i \setminus V(x_i^{-1}x_j) \subseteq U_i$ we have that $\mathcal{O}_{U_1}(U_{12}) = \mathbb{C}[x_1^{-1}x_2, x_2^{-1}x_1] = \mathcal{O}_{U_2}(U_{21})$ and therefore we have an isomorphism $U_{12} \cong U_{21}$. The schemes U_i can be glued along the intersections U_{ij} and by the Glueing lemma [Liu02, Chapter 2, Lemma 3.33] we obtain a scheme which is isomorphic to $\mathbb{P}^1$.

On the other hand, note that $\mathbb{C}[x_1, x_2]_{(x_i)} = \mathbb{C}[x_i^{-1}x_j]$. In view of the previous theorem, this gives an isomorphism from Proj($\mathbb{C}[x, y]$) with the projective space $\mathbb{P}^1$.

As with the relative Spec of quasi-coherent $\mathcal{O}_M$-algebras introduced in A.10, there exist a relative version of Proj for $\mathcal{O}_M$-algebras which we are about to discuss. The philosophy of this construction is quite similar to the one of the relative Spec and that is to construct a space by coherently glueing prescribed schemes.

Definition A.15. Let M be a scheme. A *graded $\mathcal{O}_M$-algebra* $\mathcal{B}$ is a quasi-coherent sheaf of $\mathcal{O}_M$-algebras with a grading $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{B}_d$ where $\mathcal{B}_d$ is a quasi-coherent $\mathcal{O}_M$-sub-module for all d. This means that for any affine open subset U of M, $\mathcal{B}(U) = \bigoplus_{d \geq 0} \mathcal{B}_d(U)$ is a graded $\mathcal{O}_M(U)$-algebra.

Example A.16. Let $E \to M$ be a rank r vector bundle over M and let us denote with the same symbol its sheaf of sections. This sheaf is a locally free sheaf of rank r so

$$\mathcal{B} = \bigoplus_{d \geq 0} E^{\otimes d}$$

is naturally a graded $\mathcal{O}_M$-algebra.

Theorem A.17. Let M be a scheme and $\mathcal{B}$ a graded $\mathcal{O}_M$-algebra. Then there exists a unique scheme $\mathbf{Proj}(\mathcal{B})$ and a unique morphism $\pi : \mathbf{Proj}(\mathcal{B}) \to M$ such that for every affine open subset $U \subseteq M$ we have $\pi^{-1}(U) \cong \mathrm{Proj}(\mathcal{B}(U))$ and for every inclusion $U \hookrightarrow V$ of open affines on M, the morphism $\pi^{-1}(U) \hookrightarrow \pi^{-1}(V)$ corresponds to the restriction morphism $\mathcal{B}(V) \to \mathcal{B}(U)$ [Liu02, Chapter 8, Lemma 1.8].

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