Counting absolutely indecomposable quiver representations

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Merceditas, tiempos difíciles son. Quisiera que hubieran más dedicatorias para ti, prometo esforzarme para que así sea. También, quiero que este trabajo sea un homenaje para tu hija, Nelly. Estas "pérdidas" (sí, entre comillas), me hacen ser más consciente de que somos quienes somos y llegamos donde llegamos gracias a las personas que nos rodean y amamos. Nuestros seres queridos no se pierden pues nuestros espíritus viven de ellos: de lo que nos dieron, de las enseñanzas que nos dejaron y de lo que sus vidas y ejemplos representan para nosotros.

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Contents

Introduction

Quivers are simple mathematical objects: finite directed graphs. A representation of a quiver assigns a vector space to each vertex, and a linear map to each arrow of the graph. These objects were originally introduced to treat problems of linear algebra, but it soon turned out that they play an important role in representation theory and in algebraic geometry. For instance, it was shown by M. Reineke that every projective variety is a quiver grassmannian [Rei13].

As with the classical representation theory, a quiver representation can be decomposed, up to reordering, as the direct sum of *indecomposable* representations (those that cannot be written further as direct sum of representations). Therefore, if one is to know all the representations of a given quiver, it is enough to understand its indecomposable representations. When the ground field is assumed to be finite, say the field $\mathbb{F}_q$, one can study a particular type of indecomposable representations: the *absolutely indecomposable* ones which are those indecomposable representations which remain indecomposable when we look at them as representations over $\overline{\mathbb{F}}_q$. In the 1980's, V. Kac showed that the number of absolutely indecomposable representations with fixed dimension vector (i.e. the dimensions of the vector spaces that we put at each vertex are fixed), for a given quiver without loops, is a polynomial in $\mathbb{Z}[q]$. He went further by conjecturing that the coefficients of this polynomial where actually positive.

For almost twenty years, no progress was made in solving Kac's conjecture until 2004 when W. Crawley-Boevey and M. Van den Bergh gave a proof when the dimension vector is assumed to be indivisible, that is, when the greatest common divisor of its entries is one [CBVdB04]. In 2013 a full proof of the conjecture was given by T. Hausel, E. Letellier and F. Rodríguez Villegas [HLRV13]. The ideas and techniques introduced in these works, has laid the ground and inspired tremendous breakthroughs in mathematics such as the work by O. Schiffman regarding the Poincaré polynomial of the moduli space of Higgs bundles over a compact Riemann surface [Sch16].

In this work we will focus on making a clear picture on the techniques introduced by Crawley-Boevey and Van den Bergh to count absolutely indecomposable representations and deduce Kac's conjecture in the particular setting just explained. The facts presented here are well known but we gave ourselves the task of writing an expository account of these, providing precise references whenever it cannot be self-contained.

Let us now outline the structure of this work and comment a little on the strategy of Crawley-Boevey and Van den Bergh. In the first chapter we overview the basic theory of quivers and its representations. In the second chapter, we recollect the main facts of Geometric Invariant Theory (GIT) that allow us, at the end of the chapter, to construct moduli spaces of quiver representations as linearised GIT quotients. In the third chapter we introduce representations or modules over a deformation of the path algebra of a quiver and obtain a smooth variety X_λ from the quotients obtained in the second chapter. The number of $\mathbb{F}_q$-points of this variety is deeply related to the desired count of absolutely indecomposable representations. In fact, if we know the number of rational points of X_λ then we know the desired number of absolutely indecomposable representations. Unfortunately, there is no much more information we can obtain from this variety so we introduce a one-parameter family whose generic fiber is equal to X_λ. In the fourth and last chapter, the purpose of this one-parameter family becomes clear when we see that the number of $\mathbb{F}_q$-points of both its generic and its special fiber, X_0, coincide. What is more, we see that, as the special fiber satisfies good properties such as being cohomologically pure, it is possible to reduce the count of the rational points of X_0, and hence of X_λ, to a formula depending on the dimension of the etale cohomology groups of X_0. Finally, we rewrite this formula in terms of the singular complex cohomology groups of the some sort of "complexification" of the special fiber.

Chapter 1

Quivers

In this chapter, we present fundamental notions and results on quivers and its representations which will be the framework of this memoir. Throughout this discussion we fix a field k.

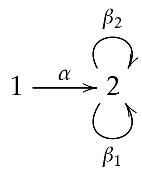
1.1 Basic definitions

Definition 1.1. A *quiver* is a finite directed graph, that is, a tuple $Q = (V, E, h, t)$ where V is a finite set of vertices, E is a set of edges and $h, t : E \to V$ maps assigning to each edge its corresponding head and tail.

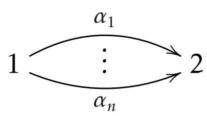
Example 1.2. The following graph represents a quiver having only one edge and vertex. It is known as the *Jordan quiver*.

Note that $h(\alpha) = t(\alpha) = 1$.

Example 1.3. For the quiver represented by the graph below, one has that $t(\alpha) = 1$ and $h(\alpha) = h(\beta_i) = t(\beta_i) = 2$ for $i = 1, 2$.



Example 1.4. The n-edges *Kronecker quiver* is the quiver having two vertices, $\{1, 2\}$, and n-edges, $\alpha_1, \ldots, \alpha_n$ such that $t(\alpha_i) = 1$ and $h(\alpha_i) = 2$ for all $i = 1, \ldots, n$ as depicted in the graph below.



Example 1.5. Given a quiver Q, we define its *double quiver*, denoted by $\overline{Q}$, to be the quiver having the same vertices as Q but set of edges given by $\overline{E} = \{\alpha, \alpha^*\}_{\alpha \in E}$ where $\overline{h}(\alpha^*) = t(\alpha)$ and $\overline{t}(\alpha^*) = h(\alpha)$. Similarly, the *opposite quiver*, Q^{op}, has the same vertices as Q and set of edges given by $E^{op} = \{\alpha^*\}_{\alpha \in E}$.

The latter is an example that we will study in greater detail later.

Definition 1.6. A *non-trivial path* in a quiver Q is a sequence $\alpha_1, \dots, \alpha_k$ of edges such that $t(\alpha_i) = h(\alpha_{i+1})$ for all $i = 1, \dots, k$. On the other hand, the *trivial path*, for each vertex $v \in V$ of Q, is the path that starts and terminates at v. We denote this path by e_v.

Definition 1.7. The *path algebra* kQ is the k-algebra generated by the edges of the quiver Q. The product of two edges or paths x, y in this algebra is defined to be:

$$xy = \begin{cases} \text{concatenation of paths} & \text{if } h(y) = t(x), \\ 0 & \text{otherwise.} \end{cases}$$

Here, $h(x)$ and $t(x)$ denote, respectively, the head and the tail of the path x. That is, the vertices where the path x ends and begins respectively.

Remark 1.8. A basis of kQ as a k-vector space is given by the set of all paths in the quiver Q.

Example 1.9. Let Q be the Jordan quiver in Example 1.2. Note that $k[x] \cong kQ$ via the unique algebra morphism sending x to α.

Example 1.10. Let Q be the quiver represented by the graph:

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3.$$

Then the basis for the k-vector space underlying kQ is given by the set $\{e_1, e_2, e_3, \alpha_1, \alpha_2, \alpha_2\alpha_1\}$.

Proposition 1.11. The following are the main properties of the path algebra kQ:

1. The trivial paths are orthogonal idempotents, that is, $e_v e_w = 0$ for $v \neq w$ and $e_v^2 = e_v$.
2. kQ has an identity given by
$$1 = \sum_{v \in V} e_v.$$
3. kQ is a finite-dimensional k-vector space if and only if Q does not contain oriented cycles. That is, if there is no path p of positive length such that $h(p) = t(p)$.
4. The vector subspaces $kQe_w, e_v kQ$ and $e_v kQ e_w$ have as bases the paths starting at v and/or terminating at w.
5. $kQ = \bigoplus_{v \in V} kQe_v$ so that each kQe_v is a projective left kQ-module.
6. For any kQ-module M, $\mathrm{Hom}_{kQ}(kQe_v, M) \cong e_v M$.

1.2 Quiver representations

We now introduce representations of quivers and state the relation between them and kQ-modules.

Definition 1.12. A k-representation of a quiver Q is a tuple $W := ((W_v)_{v \in V}, (\varphi_\alpha)_{\alpha \in E})$ where:

- For all $v \in V$, W_v is a finite dimensional vector space over k.

- For all $\alpha \in E$, $\varphi_\alpha : W_{t(\alpha)} \to W_{h(\alpha)}$ is a k-linear map.

Definition 1.13. The *dimension vector* of a representation is the tuple $d = (d_v := \dim W_v)_{v \in V} \in \mathbb{N}^{|V|}$.

We have a notion of morphism between representations of a fixed quiver, so that we can think of them as a category.

Definition 1.14. Let W, W' be two representations of a quiver Q. A morphism $f : W \to W'$ is given by a tuple $(f_v : W_v \to W'_v)_{v \in V}$ such that the diagram

$$\begin{array}{ccc} W_{t(\alpha)} & \xrightarrow{\varphi_\alpha} & W_{h(\alpha)} \\ {\scriptstyle f_{t(\alpha)}}\downarrow & & \downarrow{\scriptstyle f_{h(\alpha)}} \\ W'_{t(\alpha)} & \xrightarrow[\varphi'_\alpha]{} & W'_{h(\alpha)} \end{array}$$

commutes for all $\alpha \in E$.

There are natural notions of direct sums, subrepresentations, kernels and cokernels in the category of k-representations of a quiver Q, $\mathrm{Rep}(Q)$. Indeed, this is actually an abelian category [Kir16, Section 1.1].

Definition 1.15. Let $W, W' \in \mathrm{Rep}(Q)$, then:

- We define the *direct sum* $W \oplus W' \in \mathrm{Rep}(Q)$ by $(W \oplus W')_v := W_v \oplus W'_v$ for all $v \in V$ together with the obvious definition for the linear maps associated to each edge.
- We say that W' is a *subrepresentation* of W if $W'_v \subseteq W_v$ is a vector subspace for all $v \in V$ such that $\varphi_\alpha(W'_{t(\alpha)}) \subseteq W'_{h(\alpha)}$ for all edge $\alpha \in E$.
- If $f : W' \to W$ is a morphism of representations, then we define the subrepresentations $\ker f$ and $\mathrm{im} f$ corresponding to the kernel and image respectively of the maps $f_v : W'_v \to W_v$.
- If W' is a subrepresentation of W, we define the *quotient representation*, W/W', as $(W/W')_v := W_v/W'_v$ together with the obvious definition for the linear maps associated to each edge.

The following lemma shows us that we can identify quiver representations and kQ-modules.

Lemma 1.16. [CB92, Chapter 1] The category $\mathrm{Rep}(Q)$ is equivalent to the category of kQ-modules.

Proof. We will only discuss how to associate a kQ-module to a given quiver representation and viceversa. For a quiver representation W, we will endow $\mathcal{W} = \bigoplus_{v \in V} W_v$ with structure of kQ-module. We set $\alpha_1 \ldots \alpha_k x := \iota_{h(\alpha_1)} \circ \varphi_{\alpha_1} \circ \ldots \circ \varphi_{\alpha_k} \circ \pi_{t(\alpha_k)}(x)$ with $\iota_v : W_v \hookrightarrow \mathcal{W}$ the canonical inclusion and $\pi_v : \mathcal{W} \to W_v$ the canonical projection. On the other hand, if $\mathcal{W}$ is a kQ-module, define a representation of Q with $W_v = e_v \mathcal{W}$ and $\varphi_\alpha(x) = e_{h(\alpha)} \alpha x \in W_{h(\alpha)}$. $\square$

Many notions of the classical representation theory such as simplicity, semisimplicity and the existence of filtrations can be translated to the quiver representation setting as we will see now and in the forthcoming chapter.

Definition 1.17. A quiver representation is said to be *simple* if it does not have a non-trivial subrepresentation, *semisimple* if it can be written as the direct sum of simple representations, *indecomposable* if it cannot be written as the direct sum of non-trivial subrepresentations and *absolutely indecomposable* if it remains indecomposable if we look at it as a $\overline{k}$-representation.

Example 1.18. Consider the one edge Kronecker quiver (see Example 1.4). Note that the representations $W_s = (W_1 = k, W_2 = 0)$ and $W_s' = (W_1 = 0, W_2 = k)$ are simple. However, the representation $W_i = ((W_1 = k, W_2 = k), (\varphi_\alpha = \mathrm{Id}))$ is indecomposable. Otherwise, we would have $W_i = W_s \oplus W_s'$ which is impossible since W_s is not even a subrepresentation of W_i.

The following lemma characterizes indecomposable kQ-modules or representations:

Lemma 1.19. [Bri12, Lemma 1.3.3] Given a finite-dimensional module M over the path algebra kQ, the following conditions are equivalent:

1. M is indecomposable.

2. Any kQ-endomorphism of M is either nilpotent or invertible.

3. $\mathrm{End}_{kQ}(M) = I \oplus k\mathrm{Id}_M$ where I is a nilpotent ideal.

Remark 1.20. One can show that $\mathrm{End}_{kQ}(M)$ is a local ring such that, for $k = \mathbb{F}_q$,

$$\frac{|\mathrm{End}_{kQ}(M)|}{|\mathrm{Aut}_{kQ}(M)|} = \frac{q}{q-1}.$$

For more details on this we refer the reader to [Ben98, Chapter 1] and [Hos18, Lemma 5.17].

And we have the following decomposition theorem, which is a particular case of the Krull-Schmidt theorem for Artinian modules [Ben98, Theorem 1.4.6]:

Theorem 1.21. [Bri12, Theorem 1.3.4] Any finite-dimensional representation of Q can be written as direct sum of indecomposable representations. This decomposition is unique up to reordering.

1.3 The standard resolution

The kQ-module point of view on representations of a quiver gives us the following result, known as the *standard resolution*.

Theorem 1.22. Let M be a kQ-module. Then, there is an exact sequence

$$0 \longrightarrow \bigoplus_{\alpha\in E} kQe_{h(\alpha)} \otimes_k e_{t(\alpha)}M \xrightarrow{\partial_1} \bigoplus_{v\in V} kQe_v \otimes_k e_vM \xrightarrow{\partial_0} M \longrightarrow 0$$

where $\partial_0(p \otimes x) = px$ for $p \in kQe_v$, $x \in e_v M$ and $\partial_1(p \otimes x) = p\alpha \otimes x - p \otimes \alpha x$ for $p \in kQe_{h(\alpha)}$ and $x \in e_{t(\alpha)}M$.

Proof. We will start by checking that ∂_0 is surjective. Let $x \in M$ and, by the first two properties in Proposition 1.11, observe that

$$\partial_0\left(\sum_{v\in V} e_v \otimes e_v x\right) = \sum_{v\in V} e_v x = x$$

which gives the desired surjectivity.

Secondly, we will check the exactness of the sequence in the middle. Let $p \in kQe_{h(\alpha)}$ and $x \in e_{t(\alpha)}M$. By direct computation,

$$\partial_0 \circ \partial_1(p \otimes x) = \partial_0(p\alpha \otimes x - p \otimes \alpha x) = p\alpha x - p\alpha x = 0$$

which means that $\mathrm{Im}(\partial_1) \subseteq \ker(\partial_0)$. In order to see the other inclusion, we need to work a little bit more. Observe that every element $\xi \in \bigoplus_{v\in V} kQe_v \otimes_k e_v M$ can be written uniquely as

$$\xi = \sum_{v\in V} \sum_{\substack{p \text{ path} \\ \text{such that } t(p)=v}} p \otimes x_p$$

where the $x_p \in e_{t(p)}M$ are almost all zero. We set $\deg(\xi) :=$ length of the longest path p such that $x_p \neq 0$. If p is a non-trivial path with $t(p) = v$ then we can write $p = p'\alpha$ with $\alpha \in E$ such that $t(\alpha) = v$ and p' a path starting at $h(\alpha)$. Moreover, $\partial_1(p' \otimes x_p) = p \otimes x_p - p' \otimes \alpha x_p$, where the length of the path p' is the length of p minus one. So, recursively one can see that the class of $p \otimes x_p$ modulo $\mathrm{Im}(\partial_1)$ always contains an element of degree zero. Consequently, for $\xi \in \ker(\partial_0)$ and ξ' an element of degree zero in the class of ξ modulo $\mathrm{Im}(\partial_1)$ we have

$$0 = \partial_0(\xi) = \partial_0(\xi') = \partial_0\left(\sum_{v\in V} e_v \otimes x'_{e_v}\right) = \sum_{v\in V} x'_{e_v} \in \bigoplus_{v\in V} e_v M$$

which implies that $\xi' = 0$. Thus, $\xi \in \mathrm{Im}(\partial_1)$ and $\ker(\partial_0) \subseteq \mathrm{Im}(\partial_1)$.

Finally, we need to check that ∂_1 is injective. Suppose there exists a non-zero $\xi \in \ker(\partial_1)$. This element can be written as

$$\xi = \sum_{\alpha\in E} \sum_{\substack{p, \text{ path} \\ \text{such that } t(p)=h(\alpha)}} p \otimes x_{\alpha,p}$$

where the $x_{\alpha,p} \in e_{t(\alpha)}M$ are almost all zero. Let p' the path of maximal length such that $x_{\alpha,p'} \neq 0$, then

$$\partial_1(p' \otimes x_{\alpha,p'}) = p'\alpha \otimes x_{\alpha,p'} - p' \otimes \alpha x_{\alpha,p'}.$$

Hence, $\partial_1(\xi) \neq 0$ which leads to a contradiction. □

Remark 1.23. The resolution in Theorem 1.22 is a projective resolution of the kQ-module M. Indeed, for V an arbitrary kQ-module, $kQe_v \otimes_k V$ is isomorphic to $\dim V$ copies of the projective module kQe_v.

Theorem 1.22 implies the following lemma which will be useful on the forthcoming sections.

Lemma 1.24. Let W, W' be representations of Q. Then, there is an exact sequence

$$0 \longrightarrow \operatorname{Hom}(W,W') \longrightarrow \bigoplus\nolimits_{v\in V} \operatorname{Hom}(W_v, W'_v) \longrightarrow \bigoplus\nolimits_{\alpha\in E} \operatorname{Hom}(W_{t(\alpha)}, W'_{h(\alpha)}) \longrightarrow \operatorname{Ext}^1(W,W') \longrightarrow 0.$$

Proof. Let M, M' be the kQ-modules associated to the representations W and W' respectively. We apply the contravariant functor $\operatorname{Hom}(\cdot, M')$ to the standard resolution, so we get the exact sequence

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}(M,M') \longrightarrow \operatorname{Hom}\big(\bigoplus\nolimits_{v\in V} kQe_v \otimes_k e_v M, M'\big) \longrightarrow \operatorname{Hom}\big(\bigoplus\nolimits_{\alpha\in E} kQe_{h(\alpha)} \otimes_k e_{t(\alpha)} M, M'\big) \\ &\longrightarrow \operatorname{Ext}^1(M,M') \longrightarrow \operatorname{Ext}^1\big(\bigoplus\nolimits_{v\in V} kQe_v \otimes_k e_v M, M'\big) \longrightarrow \cdots \end{aligned}$$

Since both $\bigoplus_{v\in V} kQe_v \otimes_k e_v M$ and $\bigoplus_{\alpha\in E} kQe_{h(\alpha)} \otimes_k e_{t(\alpha)} M$ are projective, the corresponding functors functors $\operatorname{Ext}^i(\bigoplus_{v\in V} kQe_v \otimes_k e_v M, \_)$ and $\operatorname{Ext}^i(\bigoplus_{\alpha\in E} kQe_{h(\alpha)} \otimes_k e_{t(\alpha)} M, \_)$ will vanish for all $i > 0$. Now, for all $v \in V$, observe that

$$\operatorname{Hom}(kQe_v \otimes_k e_v M, M') \cong kQe_v^\vee \otimes_k e_v M^\vee \otimes_{kQ} M' \cong kQe_v^\vee \otimes_k \operatorname{Hom}(kQe_v, M').$$

Moreover, Property 6 in Proposition 1.11 gives us that $\operatorname{Hom}(kQe_v, M') \cong e_v M'$. Thus,

$$\operatorname{Hom}(kQe_v \otimes_k e_v M, M') \cong e_v M^\vee \otimes_k e_v M' \cong \operatorname{Hom}(e_v M, e_v M').$$

By the equivalence of categories in Lemma 1.16 we have $\operatorname{Hom}(e_v M, e_v M') \cong \operatorname{Hom}(W_v, W'_v)$. Similar argument works for the other term in the sequence and the result follows.

□

Corollary 1.25. Let $d, d' \in \mathbb{N}^{|V|}$ be any two dimension vectors associated to the quiver Q. Then, for all d-dimensional and d'-dimensional representations of Q, W and W' respectively, we have that $\dim(\operatorname{Hom}(W,W')) - \dim(\operatorname{Ext}^1(W,W'))$ is a constant on the dimension vectors d and d'.

Proof. The exact sequence in the previous lemma yields

$$\dim(\operatorname{Hom}(W,W')) + \dim\left(\bigoplus_{\alpha\in E} \operatorname{Hom}(W_{t(a)}, W'_{h(a)})\right) = \dim\left(\bigoplus_{v\in V} \operatorname{Hom}(W_v, W'_v)\right) + \dim(\operatorname{Ext}^1(W,W')).$$

Therefore,

$$\dim(\operatorname{Hom}(W,W')) - \dim(\operatorname{Ext}^1(W,W')) = \sum_{v\in V} d_v d'_v - \sum_{\alpha\in E} d_{t(\alpha)} d'_{h(\alpha)}.$$

□

Remark 1.26. The bilinear form on $\mathbb{Z}^{|V|}$,

$$\langle d, d' \rangle_Q := \sum_{v \in V} d_v d'_v - \sum_{\alpha \in E} d_{t(\alpha)} d'_{h(\alpha)}$$

is commonly known in the literature as the *Euler form* associated to the quiver Q [Hos18, Definition 2.4].

Chapter 2

Moduli of quiver representations

For a fixed quiver Q and a fixed field k, we are interested in classifying all its k-representations with fixed dimension vector $d \in \mathbb{N}^{|V|}$, up to isomorphism. The affine space

$$\operatorname{Rep}(Q,d) := \bigoplus_{\alpha \in E} \operatorname{Mat}(d_{h(\alpha)}, d_{t(\alpha)}, k)$$

parametrices the the d-dimensional representations over k of the quiver Q. So, we will often write a representation $W \in \operatorname{Rep}(Q,d)$ just as a tuple $W = (\varphi_\alpha)_{\alpha\in E}$ where $\varphi_\alpha \in \operatorname{Mat}(d_{h(\alpha)}, d_{t(\alpha)}, k)$.

One can consider the orbit space $\operatorname{Rep}(Q,d)/\mathrm{GL}_d$ where $\mathrm{GL}_d := \bigoplus_{v\in V} \mathrm{GL}(\alpha_v, k)$ acts by conjugation. More explicitly, for $M = (M_v)_{v\in V} \in \mathrm{GL}_d$ and $\varphi = (\varphi_\alpha : k^{d_{t(\alpha)}} \to k^{d_{h(\alpha)}})_{\alpha\in E} \in \operatorname{Rep}(Q,d)$,

$$M \cdot \varphi = (M_{h(\alpha)} \varphi_\alpha M^{-1}_{t(\alpha)})_{\alpha \in E}. \tag{2.1}$$

The points of this orbit space are in bijection with the desired isomorphism classes of representations of the quiver Q (see Lemma 2.2). However is generally bad behaved in the sense that it does not have, in many cases, a suitable algebraic structure.

Example 2.1. Consider the orbit space $\operatorname{Rep}(Q,2)/\mathrm{GL}(2,\mathbb{C})$ of the two-dimensional complex representations of the Jordan Quiver, studied in Example 1.2, which corresponds to the set of 2×2 matrices modulo conjugation. If we endow this orbit space with the quotient topology, there are points which are not closed so it does not admit a structure of complex algebraic variety. To see this, we consider the image of the set

$$\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| t \in (0,1) \right\} \subseteq \operatorname{End}(\mathbb{C}^2)$$

under the canonical projection $\operatorname{Rep}(Q,2) \to \operatorname{Rep}(Q,2)/\mathrm{GL}(2,\mathbb{C})$. This is given by the orbit of the Jordan block

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

However, this point is not closed in the quotient since its preimage does not contain the point

$$\lim_{t\to 0}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

The previous discussion tells us that we have to think on quotients in a different way and not necessarily as orbit spaces. To do so, we will use Geometric Invariant Theory (GIT). But before, we will dive deeper into the geometry of the orbits of the aforementioned action of the group GL_d in $\mathrm{Rep}(Q,d)$.

2.1 Geometry of orbits

The main reference we follow in this section is [Kir16, Chapter 2] and we assume k to be an algebraically closed field. We start with the description of the orbit and stabilizers of the action at hand:

Lemma 2.2. Let $W=(\varphi_\alpha)_{\alpha\in E}\in\mathrm{Rep}(Q,d)$. Then:

1. $O_W:=\mathrm{GL}_d\cdot W=\{W'\in\mathrm{Rep}(Q,d)|W\cong W'\}$.
2. $\mathrm{Stab}_{\mathrm{GL}_d}(W)=\mathrm{Aut}_Q(W)$.

Proof. Observe that $W'=(\varphi'_\alpha)_{\alpha\in E}\in O_W$ if and only if there exists $M=\{M_v\}_{v\in V}\in\mathrm{GL}_d$ such that

$$M\cdot W=(M_{h(\alpha)}\varphi_\alpha M_{t(\alpha)}^{-1})_{\alpha\in E}=(\varphi'_\alpha)_{\alpha\in E}=W'.$$

This is equivalent to saying that $M_{h(\alpha)}\circ\varphi_\alpha=\varphi'_\alpha\circ M_{t(\alpha)}$, for all $\alpha\in E$. Then, the diagram in Definition 1.14 commutes which means that M is a morphism of d-dimensional representations of Q. Moreover, since M is invertible, $W\cong W'$ and the first part of the lemma follows.

The second statement of the lemma is a direct consequence of the previous discussion. Observe that $M\in\mathrm{Stab}_{\mathrm{GL}_d}(W)\subseteq\mathrm{GL}_d$ satisfies $M_{h(\alpha)}\varphi_\alpha=\varphi_\alpha M_{t(\alpha)}$ for all $\alpha\in E$. That is, M is an automorphism of $W\in\mathrm{Rep}(Q,d)$. □

As the action on GL_d in Rep(Q,d) is algebraic, we are particularly interested in understanding the closed orbits which are precisely the points of the GIT quotient that we will define in the next section. We start with a proposition that relates the orbits of direct sums of representations with extensions.

Proposition 2.3. Let

$$0\longrightarrow W'\longrightarrow W\longrightarrow W''\longrightarrow 0$$

be an exact sequence of representations of Q. Then $O_{W'\oplus W''}\subseteq\overline{O_W}$.

Proof. We choose an isomorphism $W \cong W' \oplus W''$ as vector spaces. Then, for all $\alpha \in E$,

$$\varphi_\alpha = \begin{pmatrix} \varphi'_\alpha & M_\alpha \\ 0 & \varphi''_\alpha \end{pmatrix}.$$

Here $\varphi_\alpha, \varphi'_\alpha$ and φ''_α are the corresponding linear maps between the source and the target vector spaces, of the representations W, W' and W'' respectively, associated to $\alpha \in E$.

We consider as well the one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{GL}_d$ given by

$$\lambda(t) = \left\{ \begin{pmatrix} t\mathrm{Id}_{\dim(W'_v)} & 0 \\ 0 & \mathrm{Id}_{\dim(W''_v)} \end{pmatrix} \right\}_{v\in V}$$

and observe that

$$\lambda(t) \cdot \varphi = \left\{ \begin{pmatrix} \varphi'_\alpha & tM_\alpha \\ 0 & \varphi''_\alpha \end{pmatrix} \right\}_{\alpha\in E}.$$

Making $t \to 0$ we can see that $W' \oplus W''$ is in the closure of O_W from which the result follows. □

Now we introduce a useful definition that will help us to state the main results of the section:

Definition 2.4. A *filtration*, $\mathcal{F}$, for a quiver representation $W \in \mathrm{Rep}(Q,d)$ is a decreasing chain of subrepresentations

$$W = W_0 \supset W_1 \supset W_2 \supset \ldots \supset W_n = \{0\}$$

and the associated *graded representation* is

$$\mathrm{gr}\ \mathcal{F} = \bigoplus W_j/W_{j+1} \in \mathrm{Rep}(Q,d).$$

Lemma 2.5. Let $W = ((W_v)_{v\in V}, (\varphi_\alpha)_{\alpha\in E}), W' \in \mathrm{Rep}(Q,d)$. Then:

1. If $W' \cong \mathrm{gr}\ \mathcal{F}$ for some filtration $\mathcal{F}$ of W, then $O_{W'} \subseteq \overline{O_W}$.
2. If the orbit $O_{W'}$ is closed, then the converse also holds. That is, if $O_{W'} \subseteq \overline{O_W}$ then $W' \cong \mathrm{gr}\ \mathcal{F}$ for some filtration $\mathcal{F}$ of W.

Proof. By induction on the length of the filtration and Proposition 2.3 we can show the first statement of the theorem.

For the converse statement, we will need a lemma which tells us that for a closed and GL_d-invariant subset $Y \subseteq \mathrm{Rep}(Q,d)$ such that $Y \cap \overline{O_W} \neq \emptyset$, there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{GL}_d$ such that $\lim_{t\to 0} \lambda(t) \cdot W \in Y$ [Kem78, Theorem 1.4]. So if we assume that $O_{W'} \subseteq \overline{O_W}$, by the aforementioned lemma and the fact that $O_{W'}$ is closed, there exist a one-parameter subgroup λ such that $\lim_{t\to 0} \lambda(t) \cdot W = W_0 = ((W_{0,v})_{v\in V}, (\varphi_{0,\alpha})_{\alpha\in E})$ for some $W_0 \cong W'$.

This one-parameter subgroup induces as well a $\mathbb{G}_m$-representation on the family of vector spaces underlying W and, by a well-known fact from algebraic group theory [Spr98, Section 3.2.13], we can decompose, for all vertex $v \in V$,

$$W_v = \bigoplus_{n \in \mathbb{Z}} W_v^n$$

where $\lambda(t)|_{W_v^n} = t^n$. Therefore, for all edge $\alpha \in E$, the linear maps $\varphi_\alpha : W_{t(\alpha)} \to W_{h(\alpha)}$ can be written as $\varphi_\alpha = \bigoplus \varphi_\alpha^{m,n}$ for $\varphi_\alpha^{m,n} : W_{t(\alpha)}^n \to W_{h(\alpha)}^m$ linear maps.

By the previous discussions, the action of $\mathbb{G}_m$ on the family $(\varphi_\alpha)_{\alpha \in E}$ is given by $\lambda(t) \cdot \varphi_\alpha = \sum t^{m-n} \varphi_\alpha^{m,n}$ so $W_0 = \lim_{t \to 0} \lambda(t) \cdot W$ exists if and only if $\varphi_\alpha^{m,n} = 0$ for all $m < n$ because otherwise $t^{m-n} \to \infty$ as $t \to 0$. This implies that the subspaces $W^{\geq k} = \{W_v^{\geq k}\}_{v \in V}$, where

$$W_v^{\geq k} = \bigoplus_{m \geq k} W_v^m,$$

define subrepresentations of Q. Thus, we get a filtration, $\mathcal{F}$, of W:

$$\cdots \supset W^{\geq -1} \supset W^{\geq 0} \supset W^{\geq 1} \supset \cdots$$

where $W^{\geq n} = 0$ for $n \gg 0$ and $W^{\geq n} = W$ for $n \ll 0$. The underlying vector spaces, associated to the corresponding graded representation, are given by

$$(\mathrm{gr}\, \mathcal{F})_v = \bigoplus_{n \in \mathbb{Z}} W_v^{\geq n} / W_v^{\geq n+1} = \bigoplus_{n \in \mathbb{Z}} W_v^n$$

for all $v \in V$. And the corresponding linear maps, attached to each edge $\alpha \in E$, will be given by $\bigoplus \varphi_\alpha^{n,n} = \lim_{t \to 0} \lambda(t) \cdot \varphi_\alpha = \varphi_{0,\alpha}$. Therefore, $\mathrm{gr}\, \mathcal{F} \cong W_0 \cong W'$ which concludes the proof. □

The following example deals with a type of filtration of particular interest to us:

Example 2.6. A filtration, $\mathcal{F}$, of $W \in \mathrm{Rep}(Q, d)$ is said to be a *composition series* if all the quotients W_j / W_{j+1} are simple representations. Its corresponding graded representation, $W^{ss} := \mathrm{gr}\, \mathcal{F}$, is called the *semisimplification* of W as it is a semisimple representation.

The Jordan-Hölder theorem asserts that any two composition series for W have the same length and that the sets of simple representations appearing in its graded representations, counted with multiplicity, are the same [Ben98, Theorem 1.14] , [Kir16, Section 2.3]. The latter means that W^{ss} does not depend on the choice of the composition series which justify the notation adopted.

With what we have learned about the action of GL_d on $\mathrm{Rep}(Q, d)$ we can now state and prove the main result of this section.

Theorem 2.7. Let $W \in \mathrm{Rep}(Q, d)$, then

1. O_W is closed if and only if W is semisimple.
2. The closure of O_W contains a unique closed orbit, namely the orbit of its semisimplification, $O_{W^{ss}}$.

Proof. We start with the proof of the first part of the theorem. By Lemma 2.5 we have that $O_{W^{ss}} \subseteq \overline{O_W}$ so if O_W is closed then W is semisimple. Now, we prove the converse so we assume W to be semisimple. If there exists a closed orbit contained in $\overline{O_W}$, by the second part of Lemma 2.5, O_W is closed. So we only need to check what happens in the case where, hypothetically, $\overline{O_W}$ does not contain closed orbits. It is a standard fact of actions of algebraic groups on vector spaces that the closure of O_W is a union of orbits [Kir16, Section 2.2], so let $O_{W'} \subsetneq \overline{O_W}$ be a non-closed orbit. Then, we can find a GL_d-invariant polynomial whose value on the semisimplifications of W and W' is equal if and only if these are isomorphic [Bri12, Lemma 2.3.3]. But, as it is continuous, such a polynomial must be constant on orbits and its closures meaning that the semisimplification of W' must coincide with W. Thus, O_W is closed.

For the second part of the theorem, let us assume that there are two closed orbits O_{W_1} and O_{W_2} contained in O_W. Then, by the second part of Lemma 2.5, we have that there must exist filtrations, $\mathcal{F}_1$ and $\mathcal{F}_2$, of some representation V, such that $W_1 = \mathrm{gr}\ \mathcal{F}_1$ and $W_2 = \mathrm{gr}\ \mathcal{F}_2$. But, since these two orbits are closed, W_1 and W_2 are semisimple representations and, therefore, by the Jordan-Hölder theorem we must have that $W_1 \cong W_2$. □

The following corollary partially explains Theorem 2.9 which, as we will see, is one of the main motivations for considering linearized GIT quotients.

Corollary 2.8. Let Q be a quiver without oriented cycles, then $\mathrm{Rep}(Q,d)$ contains a unique closed orbit, namely the orbit of the trivial representation, which lies in the closure of any other orbit.

Proof. For all $v \in V$, let S^v be the simple representation of Q such that $S^v_{v'} = k^{\delta_{vv'}}$ together with the zero morphism for every edge $\alpha \in E$. One can show that every simple representation of Q is isomorphic to S^v for some $v \in V$ [Bri12, Proposition 1.3.1]. Therefore, the origin or zero representation will be the unique point in $\mathrm{Rep}(Q,d)$ being semisimple. Moreover, every other orbit closure contains the origin's orbit. □

2.2 A few words on Geometric Invariant Theory

Since $\mathrm{Rep}(Q,d)$ is an affine variety (it can be identified with an affine k-vector space) and GL_d a reductive algebraic group, a reasonable quotient may be given by the prime spectrum of the finitely-generated algebra of GL_d-invariant polynomials (see [New78, Theorem 3.4]) on the vector space $\mathrm{Rep}(Q,d)$,

$$\mathrm{Rep}(Q,d) /\!\!/ \mathrm{GL}_d := \mathrm{Spec}(\mathcal{O}^{\mathrm{GL}_d}_{\mathrm{Rep}(Q,d)}).$$

This is a *categorical quotient* in the sense that there exist a surjective morphism $p : \mathrm{Rep}(Q,d) \to \mathrm{Rep}(Q,d) /\!\!/ \mathrm{GL}_d$, induced by the injection $\mathcal{O}^{\mathrm{GL}_d}_{\mathrm{Rep}(Q,d)} \hookrightarrow \mathcal{O}_{\mathrm{Rep}(Q,d)}$, such that every GL_d-invariant morphism of schemes $f : \mathrm{Rep}(Q,d) \to Y$ factors uniquely through p. And, moreover, one can show that the fiber above any point of the GIT quotient via p is stable under the action of GL_d and contains exactly one closed orbit [Rei08, Section 3.3]. This means, in the words of Theorem 2.7, that p sends a representation to its corresponding semisimplification.

The following theorem due to Le Bruyn and Procesi gives us a better understanding of the algebra $\mathcal{O}_{\text{Rep}(Q,d)}^{\text{GL}_d}$ and therefore of the categorical quotient in consideration.

Theorem 2.9. [LBP90, Theorem 1] The algebra of invariants $\mathcal{O}_{\text{Rep}(Q,d)}^{\text{GL}_d}$ is generated by traces of oriented cycles in the quiver Q.

So, if the quiver Q has no oriented cycles, $\mathcal{O}_{\text{Rep}(Q,d)}^{\text{GL}_d} = k$ and $\text{Rep}(Q,d) /\!\!/ \text{GL}_d = \{\text{point}\}$. This theorem shows as well that the categorical quotient may often be reduced to a point which is not geometrically interesting. Instead, A.King constructed a richer quotient by imposing a stability condition to the representations we are classifying [Kin94]. In the reminder of this chapter we will outline King's approach.

Let X be an affine k-variety, that is, a reduced and separated affine scheme of finite type over the field k. Assume that G is a reductive algebraic group acting on X and let $\chi : G \to \mathbb{G}_m$ be an algebraic group homomorphism usually called *character*. We have an action of G on the trivial line bundle over the variety X, $L := X \times \mathbb{A}_k^1$ given by $g(x,z) = (g \cdot x, \chi(g^{-1})z)$. This action extends to one on the algebra of regular functions of L, $\mathcal{O}_L := \mathcal{O}_X \otimes_k k[z] \cong \mathcal{O}_X[z]$, which preserves its natural grading. Indeed, for any $g \in G$, and $fz^j \in \mathcal{O}_L$ a homogeneous component, $g \cdot (fz^j) = g \cdot f(\chi(g)z)^j$ which is again a homogeneous component of degree j. Moreover, fz^j will be G-invariant if and only if $g \cdot f(\chi(g)z)^j = fz^j$. Thus, we have the graded decomposition

$$\mathcal{O}_L^G = \bigoplus_{j \geq 0} \mathcal{O}_X^{\chi^j}$$

for $\mathcal{O}_X^{\chi^j} = \{f \in \mathcal{O}_X |\ f(gx) = \chi(g)^j f(x) \text{ for all } x \in X \text{ and } g \in G\}$ and we can consider the quasi-projective scheme

$$X /\!\!/_\chi G := \text{Proj}(\mathcal{O}_L^G).$$

The GIT quotient that we have just constructed, by a *G-linearization* or *linearization* of the action of G on X, has a more interesting geometry as the following example intends to show.

Example 2.10. Let $\mathbb{G}_m$ act on $\mathbb{A}_\mathbb{C}^2$ by $t(x,y) = (tx, ty)$ and $p(x,y) \in \mathbb{C}[x,y]$ a $\mathbb{G}_m$-invariant polynomial. Then, as the induced action on polynomials preserves the grading, we must have that

$$t \cdot p_d(x,y) = p_d(t^{-1}x, t^{-1}y) = t^{-d} p_d(x,y) = p_d(x,y)$$

for p_d the homogeneous component of degree d of p. This implies that p must be constant and therefore

$$\mathbb{A}_\mathbb{C}^2 /\!\!/ \mathbb{G}_m = \text{Spec}(\mathbb{C}) = \{\text{point}\}.$$

Now we consider the identity morphism $\chi : \mathbb{G}_m \to \mathbb{G}_m$. Thus,

$$\mathbb{A}_\mathbb{C}^2 /\!\!/_\chi \mathbb{G}_m = \text{Proj}(\mathcal{O}_X) = \text{Proj}(\mathbb{C}[x,y]) = \mathbb{P}_\mathbb{C}^1.$$

To better understand the GIT quotient at hand we introduce the following definitions:

Definition 2.11. A point $x \in X$ is called *χ-semistable* if there exists $d \geq 1$ and $f \in \mathcal{O}_X^{\chi^d}$ such that $x \in X_f := \{y \in X | f(y) \neq 0\}$. We denote the set of χ-semistable points as $X^{\chi-ss}$. If in addition, the G-orbits of X_f are closed in X_f and the stabilizer of x in G is finite, x is said to be χ-stable. We denote the set of χ-stable points as $X^{\chi-s}$. Finally, two χ-semistable points are said to be *S-equivalent* if and only if the closure of their corresponding orbits intersect in $X^{\chi-ss}$.

The theorem that we are about to state, known as the Hilbert-Mumford criterion, gives us necessary and sufficient conditions for a point to be semistable or stable when k is an algebraically closed field. Let $\lambda : \mathbb{G}_m \to G$ be a one-parameter subgroup of G. For any such subgroup and character $\chi : G \to \mathbb{G}_m$ we define $\langle \lambda, \chi \rangle$ from the equality $\chi(\lambda(t)) = t^{\langle \lambda, \chi \rangle}$.

Theorem 2.12. A point $x \in X$ is semistable, respectively stable, if and only if for any one-parameter subgroup λ such that $\lim_{t \to 0} \lambda(t) \cdot x$ exists we have $\langle \lambda, \chi \rangle \geq 0$, respectively $\langle \lambda, \chi \rangle > 0$.

As with the non-linearized case, we have an injection $\mathcal{O}_L^G \hookrightarrow \mathcal{O}_L$ which, after identifying $\mathrm{Proj}(\mathcal{O}_L)$ with X, induces a map $X \to X /\!\!/_\chi G$ which we can restrict to the, by definition, open set $X^{\chi-ss}$.

Theorem 2.13. [Dol03, Theorem 8.1] The map

$$p : X^{\chi-ss} \to X /\!\!/_\chi G$$

is a categorical quotient where any two points $x, x' \in X^{\chi-ss}$ have the same image under this map if and only if they are S-equivalent. Moreover, there exists an open subset $U \subseteq X /\!\!/_\chi G$ such that $p^{-1}(U) = X^{\chi-s}$ and the restriction of p to $X^{\chi-s}$ is a *geometric quotient*, that is, an orbit space.

Finally, it is worth mentioning that from the construction of the linearized GIT quotient we have an injection of algebras $\mathcal{O}_X^{\chi^0} = \mathcal{O}_X^G \hookrightarrow \mathcal{O}_L^G$ which in turn induces a projective morphism of schemes

$$\pi : X /\!\!/_\chi G \to X /\!\!/ G.$$

As a matter of fact, by Theorem 2.9, if Q is a quiver without oriented cycles, then $X /\!\!/_\chi G$ will be a projective variety.

2.3 The moduli space of quiver representations

The goal of the final section of this chapter is to specialize the discussions of the previous two sections to the quiver setting. First, we observe that the diagonal subgroup $\Delta = \{(t\mathrm{Id}_{d_v})_{v \in V} | \ t \in \mathbb{G}_m \subseteq \mathrm{GL}_d\} \cong \mathbb{G}_m$ acts trivially on $\mathrm{Rep}(Q, d)$ so it is contained in the stabilizer of any point. This implies that the action of GL_d factors through the group $\mathrm{G}_d := \mathrm{GL}_d / \Delta$ and thus taking a quotient of the action of GL_d is equivalent to a quotient of the action of G_d.

For $\lambda = (\lambda_v)_{v\in V} \in \mathbb{Z}^{|V|}$ consider the character $\chi_\lambda : \mathrm{GL}_d \to \mathbb{G}_m$ given by

$$\chi_\lambda((M_v)_{v\in V}) := \prod_{v\in V} (\det M_v)^{\lambda_v}.$$

Hence, if one is to induce a linearized GIT quotient from χ_λ one should ask, by the previous discussion, that $\chi_\lambda(\Delta) = 1$ or, equivalently, that $d \cdot \lambda = 0$. If we fix such a character, we will have, by Theorem 2.13, the categorical and geometric quotients

$$p_C : \mathrm{Rep}(Q,d)^{\chi_\lambda - ss} \to \mathrm{Rep}(Q,d) /\!\!/_{\chi_\lambda} \mathrm{G}_d \text{ and } p_G : \mathrm{Rep}(Q,d)^{\chi - s} \to \mathrm{Rep}(Q,d)^{\chi - s}/\mathrm{G}_d$$

respectively. The right-most spaces are called the *moduli spaces of χ_λ-semistable and χ_λ-stable quiver representations* respectively.

In Theorem 2.7, we showed that the Zariski closure of a GL_d-orbit contains a unique closed orbit, namely the orbit of its corresponding semisimplification. Then, as $\mathrm{Rep}(Q,d) /\!\!/_{\chi_\lambda} \mathrm{G}_d$ parametrizes χ_λ-semistable representations up to S-equivalence, this linearized GIT quotient actually classifies the closed orbits of the set $\mathrm{Rep}(Q,d)^{\chi_\lambda - ss}$. Moreover, if we identify $\mathrm{Rep}(Q,d) /\!\!/_{\chi_\lambda} \mathrm{G}_d$ with the aforementioned set of closed orbits, then the map p_C sends an orbit of $\mathrm{Rep}(Q,d)^{\chi_\lambda - ss}$ to the orbit of its corresponding semisimplification.

One of the key points of the construction of these moduli spaces by King was to translate the GIT stability condition into a representation-theoretic one. Let us then introduce this alternative notion of stability:

Definition 2.14. Let $\lambda \in \mathbb{Z}^{|V|}$ such that $\lambda \cdot d = 0$. A d-dimensional k-representation W of the quiver Q is *λ-semistable (stable)* if $\lambda \cdot \dim W' \geq 0$ ($\lambda \cdot \dim W' > 0$) for all proper and non-trivial k-subrepresentation $W' \subset W$.

Remark 2.15. For $\lambda \in \mathbb{Z}^{|V|}$, we define the *slope* of a representation $W \in \mathrm{Rep}(Q,d)$ as

$$\mu_\lambda(W) = \frac{\sum_{v\in V} -\lambda_v d_v}{\sum_{v\in V} d_v} \in \mathbb{Q}.$$

We say that W is semistable (stable) if $\mu_\lambda(W') \leq \mu_\lambda(W)$ ($\mu_\lambda(W') < \mu_\lambda(W)$) for all proper and non-trivial subrepresentations $W' \subseteq W$. One can show that slope λ-stability is equivalent to λ-stability as in the previous definition. We refer the reader to [Rei08, Section 4] for more details on this.

And, as one would expect:

Theorem 2.16. [Kin94, Proposition 3.1] Let $\lambda \in \mathbb{Z}^{|V|}$ such that $\lambda \cdot d = 0$. Then a representation $W \in \mathrm{Rep}(Q,d)$ is χ_λ-semistable (χ_λ-stable), if and only if W is λ-semistable(λ-stable).

Remark 2.17. For $\lambda = 0$, the previous theorem implies that a representation is λ-stable if and only if is simple. For $\lambda \neq 0$ we have that every simple representation is λ-stable.

Example 2.18. Let Q be the Kronecker quiver with n edges (see Example 1.4). For the dimension vector $d = (1,1)$ we can identify $\mathrm{Rep}(Q,d) \cong \mathbb{A}_k^n$ and the action of $\mathrm{GL}_d = \mathbb{G}_m \times \mathbb{G}_m$ on this space, following Equation 2.1, will be given by $(t_1,t_2)\cdot x = \mathrm{diag}(t_2t_1^{-1},\dots,t_2t_1^{-1})x$. Also, we note that $\mathcal{O}_{\mathrm{Rep}(Q,d)} \cong k[x_1,\dots,x_n]$ and that, by Theorem 2.9, $\mathcal{O}_{\mathrm{Rep}(Q,d)}^{\mathrm{GL}_d} = k$ as Q has no oriented cycles. Hence, $\mathrm{Rep}(Q,d) /\!\!/ \mathrm{GL}_d = \mathrm{Spec}(\mathcal{O}_{\mathrm{Rep}(Q,d)}^{\mathrm{GL}_d}) = \{\text{point}\}$.

We now study the GIT quotients and semistable loci for $\lambda_+ = (1,-1)$ and $\lambda_- = (-1,1)$. Fix a representation $x \in \mathrm{Rep}(Q,d)$ and observe that if $x \neq 0$, the non-trivial and proper subrepresentations have dimension vector $(0,1)$. On the other hand, if $x = 0$ we have two possible dimension vectors for the proper and non-trivial subrepresentations, these are $(0,1)$ and $(1,0)$. So if x is a λ_+-semistable representation, then for every proper and non-trivial subrepresentation, with dimension vector (d_1,d_2), we must have $(d_1,d_2)\cdot\lambda_+ = d_1 - d_2 \geq 0$ which is absurd. Hence, $\mathrm{Rep}(Q,d)^{\lambda_+-ss} = \varnothing$. A similar argument when x is assumed to be λ_--semistable lead us to conclude that $\mathrm{Rep}(Q,d)^{\lambda_--ss} = \mathbb{A}_k^n \setminus \{0\}$.

Finally, we will give an explicit description of the GIT quotient $\mathrm{Rep}(Q,d) /\!\!/_{\chi_{\lambda_-}} \mathrm{GL}_d$. We observe that the character $\chi_{\lambda_-} : \mathrm{GL}_d \to \mathbb{G}_m$ is given by $(t_1,t_2) \mapsto t_2/t_1$ and therefore the action of GL_d on the trivial bundle $L = \mathrm{Rep}(Q,d) \otimes \mathbb{A}_k^1$ will be given by $(t_1,t_2)\cdot(x,y) = ((t_1,t_2)\cdot x, (t_1/t_2)y)$. We have then a graded decomposition

$$\mathcal{O}_L^{\mathrm{GL}_d} \cong \bigoplus_{j\geq 0} \mathcal{O}_{\mathbb{A}_k^n}^{\chi_{\lambda_-}^j}$$

where $\mathcal{O}_{\mathbb{A}_k^n}^{\chi_{\lambda_-}^j} = \{f \in k[x_1,\dots x_n] \mid f \text{ homogeneous of degree } j\}$. Thus,

$$\mathrm{Rep}(Q,d) /\!\!/_{\chi_{\lambda_-}} \mathrm{GL}_d = \mathrm{Proj}(\mathcal{O}_L^{\mathrm{GL}_d}) = \mathrm{Proj}(k[x_1,\dots,x_n]) = \mathbb{P}_k^{n-1}$$

and the map $p_C : \mathrm{Rep}(Q,d)^{\lambda_--ss} \to \mathrm{Rep}(Q,d) /\!\!/_{\chi_{\lambda_-}} \mathrm{GL}_d$, introduced in Theorem 2.13, is just the canonical projection.

We would like to finish this chapter by summarizing, in the following diagram, the moduli spaces and the corresponding quotient maps that we have introduced along the way:

$$\begin{array}{ccccc}
\mathrm{Rep}(Q,d)^{\lambda-s} & \xhookrightarrow[\text{Open}]{} & \mathrm{Rep}(Q,d)^{\lambda-ss} & \xhookrightarrow[\text{Open}]{} & \mathrm{Rep}(Q,d) \\
\downarrow{\scriptstyle p_G} & & \downarrow{\scriptstyle p_C} & & \downarrow{\scriptstyle p} \\
\mathrm{Rep}(Q,d)^{\lambda-s}/\mathrm{G}_d & \xhookrightarrow[\text{Open}]{} & \mathrm{Rep}(Q,d) /\!\!/_{\chi_\lambda} \mathrm{G}_d & \xrightarrow[\text{Projective}]{\pi} & \mathrm{Rep}(Q,d) /\!\!/ \mathrm{G}_d.
\end{array}$$

Chapter 3

Nakajima quiver varieties

The quiver varieties we are going to deal with in this section were introduced by Nakajima with the purpose of studying some type of Yang-Mills equations on a particular class of four-dimensional manifolds called ALE spaces [Nak94]. These varieties have a rich geometry which can be studied by exploiting the canonical symplectic structure of the cotangent bundle of the affine space $\mathrm{Rep}(Q,d)$. In our context, Nakajima quiver varieties will reduce our problem at hand, which is counting absolutely indecomposable d-dimensional representations over $\mathbb{F}_q$, to count $\mathbb{F}_q$-points on these.

3.1 The moment map and the deformed preprojective algebra

From now on, $Q=(V,E,h,t)$ will be a finite quiver without loops and let $\overline{Q}$, Q^{op} its corresponding doubled and opposite quivers respectively (see Example 1.5). For $d\in\mathbb{N}^{|V|}$, we have a bijection

$$\mathrm{Rep}(\overline{Q},d)\cong\mathrm{Rep}(Q,d)\oplus\mathrm{Rep}(Q^{op},d).$$

The vector space $\mathrm{Rep}(Q^{op},d)\cong\bigoplus_{\alpha\in E}\mathrm{Mat}(d_{t(a)},d_{h(a)},k)$ can be identified with the dual of the vector space $\mathrm{Rep}(Q,d)$ via the map

$$\begin{aligned}\mathrm{Rep}(Q^{op},d)&\to\mathrm{Rep}(Q,d)^{\vee}\\(Y_{\alpha^*})_{\alpha\in E}&\mapsto((X_\alpha)_{\alpha\in E}\mapsto\textstyle\sum_{\alpha\in E}\mathrm{Tr}(X_\alpha Y_{\alpha^*})).\end{aligned}\tag{3.1}$$

Therefore,

$$\mathrm{Rep}(\overline{Q},d)\cong\mathrm{Rep}(Q,d)\oplus\mathrm{Rep}(Q,d)^{\vee}\cong T^*\mathrm{Rep}(Q,d)$$

where $T^*\mathrm{Rep}(Q,d)$ stands for the cotangent bundle of $\mathrm{Rep}(Q,d)$. In what follows, $\pi:T^*\mathrm{Rep}(Q,d)\to\mathrm{Rep}(Q,d)$ will be the canonical projection onto the first factor.

We consider as well the map

$$\begin{aligned}\mu:\mathrm{Rep}(\overline{Q},d)&\to\mathfrak{g}_d\\(X_\alpha,X_{\alpha^*})_{\alpha\in E}&\mapsto\textstyle\sum_{\alpha\in E}[X_\alpha,X_{\alpha^*}]\end{aligned}\tag{3.2}$$

for $\mathfrak{g}_d = \{(M_v)_{v\in V} \in \mathfrak{gl}_d | \sum_{v\in V} \mathrm{Tr}(M_v) = 0\}$ the Lie algebra of the algebraic group G_d (see Section 2.3) and $\mathfrak{gl}_d$ the lie algebra of GL_d. This map can be regarded as a *moment map* with respect to the symplectic structure of the cotangent bundle $T^*\mathrm{Rep}(Q,d)$ given by the differential form

$$\omega(X,Y) = \sum_{\alpha\in E} \mathrm{Tr}(X_\alpha Y_{\alpha^*} - X_{\alpha^*} Y_\alpha) \tag{3.3}$$

where $X = (X_\alpha, X_{\alpha^*})_{\alpha\in E}, Y = (Y_\alpha, Y_{\alpha^*})_{\alpha\in E}$ are points in $T^*\mathrm{Rep}(Q,d) \cong \mathrm{Rep}(\overline{Q},d)$ [Hos18, Section 4.1].

Remark 3.1. For the sake of simplicity, we denoted in Equation 3.2,

$$\sum_{\alpha\in E}[X_\alpha, X_{\alpha^*}] := \left(\sum_{\substack{\alpha\in E\\ h(\alpha)=v}} X_\alpha X_{\alpha^*} - \sum_{\substack{\alpha\in E\\ t(\alpha)=v}} X_{\alpha^*} X_\alpha \right)_{v\in V}.$$

Our goal now is to understand the fibers of the moment map. Recall the path algebra $k\overline{Q}$ and the trivial path e_v introduced, respectively, in Definition 1.6 and Definition 1.7, and let $\lambda \in k^{|V|}$. The quotient

$$\Pi^\lambda = k\overline{Q} \Big/ \left(\sum_{\alpha\in E}[\alpha,\alpha^*] - \sum_{v\in V} \lambda_v e_v \right).$$

is called the *deformed preprojective algebra*. By Lemma 1.16, the category of Π^λ-modules is equivalent to the category of representations $((W_v)_{v\in V}, (\varphi_\alpha, \varphi_{\alpha^*})_{\alpha\in E})$ of the doubled quiver $\overline{Q}$ such that

$$\sum_{\substack{\alpha\in E\\ h(\alpha)=v}} \varphi_\alpha \varphi_{\alpha^*} - \sum_{\substack{\alpha\in E\\ t(\alpha)=v}} \varphi_{\alpha^*} \varphi_\alpha = \lambda_v \mathrm{Id}_{W_v} \tag{3.4}$$

for all $v \in V$. In particular, note that

$$\mathrm{Tr}\left(\sum_{\alpha\in E}[\varphi_\alpha, \varphi_{\alpha^*}] \right) = \mathrm{Tr}\left(\sum_{v\in V} \lambda_v \mathrm{Id}_{W_v} \right) = 0.$$

The previous discussions can be summarized in the following lemma:

Lemma 3.2. [CBVdB04, Lemma 2.1.1] If $\lambda \in k^{|V|}$ is such that $\lambda\cdot d \neq 0$ then $\mu^{-1}(\lambda) = \mathrm{Rep}(\Pi^\lambda, d) = \varnothing$.

Remark 3.3. In the previous lemma we identified tht tuple $(\lambda_v \mathrm{Id}_{W_v})_{v\in V}$ with the vector $\lambda \in k^{|V|}$ and $\mathrm{Rep}(\Pi^\lambda, d)$ with the subvariety of $\mathrm{Rep}(\overline{Q},d)$ given by representations satisfying Equation 3.4 for all $v \in V$.

Our goal is to count absolutely indecomposable representations of Q with dimension vector d over $\mathbb{F}_q$ so at this point, is natural to ask how do the double quiver $\overline{Q}$ and the deformed preprojective algebra Π^λ fit in this picture. The following proposition is a first step towards answering this question.

Proposition 3.4. [CB01, Theorem 3.3] Let k be an algebraically closed field and $W = (\varphi_\alpha)_{\alpha\in E} \in \mathrm{Rep}(Q,d)$. Then $W \in \pi(\mathrm{Rep}(\Pi^\lambda,d)) = \pi(\mu^{-1}(\lambda))$ if and only if the dimension vector, $d' \in \mathbb{Z}^{|V|}$, of any given direct summand of W satisfies $\lambda\cdot d' = 0$. In this case, $\pi^{-1}(W) = \mathrm{Ext}^1(W,W)^\vee$.

Proof. We start by taking $W' = W$ in Lemma 1.24. After dualizing and identifying, via the trace pairing in Equation 3.1, $\mathrm{Rep}(Q^{op},d) \cong \mathrm{Rep}(Q,d)^\vee$ and $\mathfrak{gl}_d \cong \mathfrak{gl}_d^\vee$ we get the exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1(W,W)^\vee \longrightarrow \mathrm{Rep}(Q^{op},d) \xrightarrow{\tilde{\partial}_1} \mathfrak{gl}_d \xrightarrow{\tilde{\partial}_0} \mathrm{End}(W)^\vee \longrightarrow 0.$$

We can be more explicit in describing $\tilde{\partial}_0$ and $\tilde{\partial}_1$. By the equivalence of categories described in Lemma 1.16 we can see that

$$\tilde{\partial}_1((\varphi_{\alpha^*})_{\alpha\in E}) = \sum_{\alpha\in E}[\varphi_\alpha,\varphi_{\alpha^*}] \text{ and } \tilde{\partial}_0((\theta_v)_{v\in V}) = \left((\phi_v)_{v\in V} \mapsto \sum_{v\in V}\mathrm{Tr}(\theta_v\phi_v)\right)$$

for all $(\varphi_{\alpha^*})_{\alpha\in E} \in \mathrm{Rep}(Q^{op},d)$ and $(\theta_v)_{v\in V} \in \mathfrak{gl}_d$. In particular, this implies that if $W \in \pi(\mu^{-1}(\lambda))$ then $\sum_{v\in V}\lambda_v\mathrm{Tr}(\phi_v) = 0$ for all $(\phi_v)_{v\in V} \in \mathrm{End}(W)$.

With the previous discussion in mind, suppose that $W \in \pi(\mu^{-1}(\lambda))$ has a direct summand with dimension vector $d' \in \mathbb{N}^{|V|}$. Then, for $(\phi_v)_{v\in V} \in \mathrm{End}(W)$ the morphism given by the projection onto this summand, we have $0 = \sum_{v\in V}\lambda_v\mathrm{Tr}(\phi_v) = \lambda\cdot d'$ which proves one side of the implication.

For the other direction, by Theorem 1.21 it is sufficient to check the statement for W indecomposable whose dimension vector, $d \in \mathbb{Z}^{|V|}$, satisfies $d\cdot\lambda = 0$. We recall that, by Lemma 1.19, all $\phi_v : W_v \to W_v$ in $(\phi_v)_{v\in V} \in \mathrm{End}(W)$ is the sum of a multiple of the identity and a nilpotent element. More explicitly, there exists $t \in k$ such that for all $v \in V$, $\phi_v = t\mathrm{Id}_{\dim(W_v)} + N_v$ with N_v a nilpotent matrix. Thus, by considering λ as an element in $\mathfrak{gl}_d$, we can see that $\tilde{\partial}_0(\lambda) = 0$ and, by exactness of the sequence above, it is in the image of $\tilde{\partial}_1$. This implies that there exists $W' \in \mathrm{Rep}(Q^{op},d)$ such that $\tilde{\partial}_1(W') = \lambda$ and $\pi(W,W') = W$, or in other words, that $W \in \pi(\mu^{-1}(\lambda))$.

Finally, observe that $\pi^{-1}(W) = \{W' \in \mathrm{Rep}(Q^{op},d)|\tilde{\partial}_1(W') = \lambda\} = \{W' + \mathrm{Ext}^1(W,W)^\vee|\tilde{\partial}_1(W') = \lambda\}$ which gives us the last part of the proposition. □

3.2 Generic parameters

In order to further explore the relation between $\mu^{-1}(\lambda) = \mathrm{Rep}(\Pi^\lambda,d)$ and the absolutely indecomposable representations of Q, we introduce the following definition and discuss some properties regarding the fibers of the moment map μ.

Definition 3.5. $\lambda \in \mathbb{Z}^{|V|}$ is said to be *generic* with respect to the dimension vector $d \in \mathbb{N}^{|V|}$ if $\lambda\cdot d = 0$ but $\lambda\cdot\beta \neq 0$ for all $0 < \beta < d$ ($\beta \neq 0$, $\beta \neq d$ and $0 \leq \beta_v \leq d_v$ for all $v \in V$).

Remark 3.6. Such a generic λ exists if and only if d is *indivisible*, that is, if the greatest common divisor of the components of the dimension vector is 1. To see this, note that the dimension vector is not indivisible if there exist $\beta \in \mathbb{N}^{|V|}$ and $c \in \mathbb{N}$ such that $c\beta = d$. On the other hand, $\lambda \cdot d = 0$ implies that λ is orthogonal to d and therefore to β.

Remark 3.7. The latter definition implies that, over an algebraically closed field, the notions of λ-semistability and λ-stability, in the sense of Definition 2.14, coincide. That is to say, $\mathrm{Rep}(\overline{Q}, d)^{\lambda-ss} = \mathrm{Rep}(\overline{Q}, d)^{\lambda-s}$ for $\lambda \in \mathbb{Z}^{|V|}$ generic with respect to d. Indeed, if $W \in \mathrm{Rep}(\overline{Q}, d)^{\lambda-ss}$, then for all subrepresentation W' of W we must have that $\dim W' \cdot \lambda \geq 0$ but, since λ is generic, this inequality must be strict so W is actually λ-stable.

Proposition 3.8. Let $\lambda \in \mathbb{Z}^{|V|}$ be generic for the dimension vector d and $W \in \mathrm{Rep}(\overline{Q}, d)^{\lambda-s}$, then $\mathrm{End}(W) = k$.

Proof. Let μ denote the slope of W (see Remark 2.15). One can show that the subcategory of λ-semistable representations of slope equal to μ is full and abelian [Rei08, Section 4]. This means that for all quiver representation morphism $f : W \to W$ both $\ker f$ and $\mathrm{im} f$ must be λ-semistable and have slope equal to μ. However, $\mu = 0$ since λ is generic with respect to d and therefore f is either the zero morphism or an isomorphism. Finally, if f is an isomorphism, then f must be given by the multiplication by a non-zero scalar as k is algebraically closed field which concludes the proof. □

What is more, for fields of characteristic zero or sufficiently large positive characteristic we can identify the representations of the preprojective algebra, Π^λ, contained in the latter sets with the whole set $\mathrm{Rep}(\Pi^\lambda, d)$:

Lemma 3.9. [CBVdB04, Lemma 2.1.3] Let $\lambda \in \mathbb{Z}^{|V|}$ be generic with respect to the dimension vector d. Then, for a field k of characteristic zero or sufficiently large prime characteristic, we have $\mathrm{Rep}(\Pi^\lambda, d) = \mathrm{Rep}(\Pi^\lambda, d)^{\lambda-ss} = \mathrm{Rep}(\Pi^\lambda, d)^{\lambda-s}$.

Proof. First, by Lemma 3.2, observe that representations in the set $\mathrm{Rep}(\Pi^\lambda, d)^{\lambda-s}(k)$ are simple. Thus, it suffices to prove this claim after base changing to an algebraic closure of k since simple elements, hence λ-stable and λ-semistable, of $\mathrm{Rep}(\Pi^\lambda, d)(k)$ will remain simple if we see them as elements of $\mathrm{Rep}(\Pi^\lambda, d)(\overline{k})$.

If $\mathrm{Rep}(\Pi^\lambda, d')(\overline{k}) \neq \emptyset$, then $\lambda \cdot d' = 0$ holds in $\overline{k}$ by Lemma 3.2. But λ is generic with respect to d, then for all $d' < d$ we have $d' \cdot d \neq 0$ in $\overline{k}$, for $\overline{k}$ of characteristic 0 or $p \gg 0$. Therefore, any k-representation in $\mathrm{Rep}(\Pi^\lambda, d)(\overline{k})$ is λ-semistable and λ-stable as it has no subrepresentations. □

Remark 3.10. One can use the same argument to see that the last lemma holds for every parameter $t \neq 0 \in k^{|V|}$ in the line joining 0 and λ.

One of the main reasons we are interested in indivisible dimension vectors and generic parameters is the following:

Lemma 3.11. Let $\lambda \in \mathbb{Z}^{|V|}$ be generic with respect to the dimension vector $d \in \mathbb{Z}^{|V|}$. Then:

1. For the restricted projection map

$$\pi : \mathrm{Rep}(\Pi^\lambda, d) \to \mathrm{Rep}(Q, d)$$

   we have that $\pi(\mathrm{Rep}(\Pi^\lambda, d))$ consist of indecomposable representations.

2. The set of k-points of this image is equal to the subset of absolutely indecomposable k-representations of $\mathrm{Rep}(Q, d)$.

Proof. For the first part of the lemma we assume, by contradiction, that $W \in \pi(\mathrm{Rep}(\Pi^\lambda, d))$ is decomposable, then, by Proposition 3.4, for every direct summand W' of W we must have that $\dim W' \cdot \lambda = 0$ which contradicts the genericity of λ.

Now we prove the second statement of the lemma. Let $W \in \mathrm{Rep}(Q, d)$ be an absolutely indecomposable k-representation. By Definition, this means that if we look at W as a representation over $\overline{k}$, it is still indecomposable (see Definition 1.17). Hence, by Proposition 3.4, W belongs to $\pi(\mathrm{Rep}(\Pi^\lambda, d))(k)$. Conversely, let W be a k-point of $\pi(\mathrm{Rep}(\Pi^\lambda, d))$, then, by the first statement of this lemma, W is indecomposable as a representation over $\overline{k}$. Assume by contradiction that W is decomposable as a k-representation then, by scalar extension, W will be decomposable as a $\overline{k}$-representation which is absurd. □

3.3 One-parameter families

Let $\lambda \in \mathbb{Z}^{|V|}$ generic with respect to the dimension vector d and $L \cong \mathbb{A}^1_k$ be the line joining 0 and λ. We consider the family of fibres of the moment map (see Equation 3.2) over L and define

$$\Xi := \mu^{-1}(L) \cap \mathrm{Rep}(\overline{Q}, d)^{\lambda-ss} /\!\!/_{\chi_\lambda} \mathrm{G}_d.$$

The moment map induces a G_d-equivariant application $\mu^{-1}(L) \to L$ which in turn gives a morphism $f : \Xi \to L$ such that

$$X_t := f^{-1}(t) = \mu^{-1}(t) \cap \mathrm{Rep}(\overline{Q}, d)^{\lambda-ss} /\!\!/_{\chi_\lambda} \mathrm{G}_d.$$

We will refer to the variety X_0 as the *special fiber* and, for $t \neq 0$, we will call X_t the *generic fiber*.

Lemma 3.12. There exists a non-empty open subset $U \subseteq \mathrm{Spec}\,\mathbb{Z}$ such that the family $f : \Xi|_U \to L|_U$ is smooth [CBVdB04, Lemma 2.1.3].

Proof. Here we think of Ξ and L as schemes over $\mathbb{Z}$ rather than over the field k. Indeed, what we discuss in Section 4.4 remains true for schemes over $\mathbb{Z}$. So, by the spreading out theorem, it is enough to check smoothness after base changing to $\overline{\mathbb{Q}}$ [Poo17, Theorem 3.2.1]. We first recall, by Proposition 3.8, that every point $W \in \mathrm{Rep}(\Pi^\lambda, d)$ is a simple representation. Therefore, by Lemma 2.2, $\mathrm{End}(W)(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$ and $\mathrm{Stab}_{\mathrm{G}_d(\overline{\mathbb{Q}})}$ is trivial.

Now we recall a standard fact of symplectic geometry which reads that if the differential at W of the G_d action, which is a linear map between tangent spaces $\mathfrak{g}_d \to T^*_W\mathrm{Rep}(\overline{Q}, d)$, is an injection then the moment map μ is smooth at W [CBVdB04, Lemma 2.1.5]. So the discussion in the previous paragraph implies that the moment map is indeed smooth at W [Bor91, Chapter II, Proposition 6.7] and, therefore, in the whole set $\mathrm{Rep}(\overline{Q}, d)^{\lambda-ss} \cap \mu^{-1}(L)$.

From above, note that the map $\mathrm{Rep}(\overline{Q}, d)^{\lambda-ss} \cap \mu^{-1}(L) \to L$ must be also smooth. And, on the other hand, observe that the geometric quotient $p_G : \mathrm{Rep}(\overline{Q}, d)^{\lambda-ss} \cap \mu^{-1}(L) \to \Xi$ (see Section 2.3) is smooth given that the action of the group G_d on this set is free [Rei08, Proposition 3.5]. Finally, since the diagram

$$\begin{array}{ccc} \mathrm{Rep}(\overline{Q}, d)^{\lambda-ss} \cap \mu^{-1}(L) & \xrightarrow{p_G} & \Xi \\ & \searrow & \downarrow f \\ & & L \end{array}$$

is commutative, we must have surjectivity on tangent spaces on the map $f : \Xi \to L$ and hence, the desired smoothness. ☐

Corollary 3.13. For fields of characteristic zero or sufficiently large positive characteristic, the special fiber, X_0, is smooth.

Proof. The proper closed subsets of $\mathrm{Spec}(\mathbb{Z})$ are given by finite unions of non-zero prime ideals of $\mathbb{Z}$ so that every open set contains the generic point given by the zero prime ideal. Thus, the fibers of a morphism of schemes $g : X \to U$, with $U \subset \mathrm{Spec}(\mathbb{Z})$ open, are $X \otimes_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Q})$ above the generic point and $X \otimes_{\mathbb{Z}} \mathrm{Spec}(\mathbb{F}_p)$ above every $p\mathbb{Z} \in U$ with $p \neq 0$.

As base change commutes with the formation of GIT quotients and taking semistable open sets with respect to a linearization (see Section 4.4) we then have the desired result. ☐

Corollary 3.14. For all $t \in L$, the dimension of the variety X_t is

$$\dim(X_t) = 2 - 2\langle d, d\rangle_Q.$$

Proof. From Section 2.3 and Remark 3.7 we may identify $X_t = (\mu^{-1}(t) \cap \mathrm{Rep}(\overline{Q}, d)^{\lambda-ss})/\mathrm{G}_d$. Note that

$$\dim(X_t) = \dim(\mu^{-1}(t) \cap \mathrm{Rep}(\overline{Q}, d)^{\lambda-ss}) - \dim(\mathrm{G}_d) = \dim(\mu^{-1}(t)) - \dim(\mathrm{G}_d).$$

Here, we have appealed to the fact that X_t is smooth, the G_d-action is free and the dimension of the open subvariety $\mu^{-1}(t) \cap \mathrm{Rep}(\overline{Q}, d)^{\lambda-ss}$ is the same as the dimension of $\mu(t)$ [Liu02, Proposition 5.19]. Now, observe that $\dim(\mathrm{G}_d) = \dim(\mathrm{GL}_d) - \dim(\Delta) = \dim(\mathrm{GL}_d) - 1$. On the other hand, from the proof of Lemma 3.12 we can see that t is a regular value of the moment map, $\mu : \mathrm{Rep}(\overline{Q}, d) \to$

$\mathfrak{g}_d$, therefore, $\dim(\mu^{-1}(t)) = \dim(\mathrm{Rep}(\overline{Q}, d)) - \dim(\mathrm{G}_d) = 2\dim(\mathrm{Rep}(Q,d)) - \dim(\mathrm{G}_d)$. After combining these equalities, we get:

$$\dim(X_t) = 2(\dim(\mathrm{Rep}(Q,d)) - \dim(\mathrm{GL}_d)) + 2.$$

The desired result follows from the definition of the Euler form (see Remark 1.26). □

Theorem 3.15. [CBVdB04, Proposition 2.2.1] Fix $k = \overline{\mathbb{F}}_p$. Let $\lambda \in \mathbb{Z}^{|V|}$ be generic for the dimension vector $d \in \mathbb{N}^{|V|}$. For $p >> 0$, the number of absolutely indecomposable d-dimensional representations of the quiver Q over the field $\mathbb{F}_q$, $\mathcal{A}_{Q,d}(\mathbb{F}_q)$, is given by

$$\mathcal{A}_{Q,d}(\mathbb{F}_q) = q^{-e}|X_\lambda(\mathbb{F}_q)|$$

with $e = 1 - \langle d, d\rangle_Q = \frac{1}{2}\dim(X_\lambda)$ and $q = p^r$.

Proof. We let $\mathrm{Rep}(Q,d)^{\mathrm{a.i}}$ be the set of absolutely indecomposable representations. From Burnside's formula, see for instance [Hos18, Lemma 5.15], we have that

$$\mathcal{A}_{Q,d}(\mathbb{F}_q) = |\mathrm{Rep}^{\mathrm{a.i}}(Q,d)(\mathbb{F}_q)/\mathrm{G}_d(\mathbb{F}_q)| = \frac{1}{|\mathrm{G}_d(\mathbb{F}_q)|}\sum_{W \in \mathrm{Rep}^{\mathrm{a.i}}(Q,d)(\mathbb{F}_q)} |\mathrm{Stab}_{\mathrm{G}_d(\mathbb{F}_q)}(W)|.$$

Since $\mathrm{Stab}_{\mathrm{G}_d(\mathbb{F}_q)}(W) \cong \mathrm{Aut}(W)/\Delta$ we have that $|\mathrm{Stab}_{\mathrm{G}_d(\mathbb{F}_q)}(W)| = q^{-1}|\mathrm{End}(W)|$ by Remark 1.20. Combining this with Proposition 3.4, which still holds if the field we are working with is not algebraically closed, gives

$$\mathcal{A}_{Q,d}(\mathbb{F}_q) = \frac{q^{-1}}{|\mathrm{G}_d(\mathbb{F}_q)|}\sum_{(W,W') \in \mu^{-1}(\lambda)(\mathbb{F}_q)} \frac{|\mathrm{End}(W)|}{|\mathrm{Ext}^1(W,W)|}.$$

Now we use Corollary 1.25 to obtain

$$\mathcal{A}_{Q,d}(\mathbb{F}_q) = \frac{q^{-1}}{|\mathrm{G}_d(\mathbb{F}_q)}\sum_{(W,W') \in \mu^{-1}(\lambda)(\mathbb{F}_q)} q^{\langle d,d\rangle_Q} = q^{\langle d,d\rangle_Q - 1}\frac{|\mu^{-1}(\lambda)(\mathbb{F}_q)|}{|\mathrm{G}_d(\mathbb{F}_q)|}.$$

The last step consists in showing that

$$\frac{|\mu^{-1}(\lambda)(\mathbb{F}_q)|}{|\mathrm{G}_d(\mathbb{F}_q)|} = |X_\lambda(\mathbb{F}_q)|.$$

By Proposition 3.8, if $(W,W') \in \mu^{-1}(\lambda)$ then $\mathrm{End}(W,W') = k$ and $\mathrm{Stab}_{\mathrm{G}_d}(W,W')$ is trivial. This implies that the action of $\mathrm{G}_d(\mathbb{F}_q)$ on $\mathrm{Rep}(\Pi^\lambda, d)(\mathbb{F}_q)$ is free and therefore

$$\frac{|\mu^{-1}(\lambda)(\mathbb{F}_q)|}{|\mathrm{G}_d(\mathbb{F}_q)|} = \left|\frac{\mu^{-1}(\lambda)(\mathbb{F}_q)}{\mathrm{G}_d(\mathbb{F}_q)}\right|.$$

In what is left, we will denote by F the extension of the Frobenius automorphism $F: k \to k$, given by $x \mapsto x^q$, to G_d and $\mu^{-1}(\lambda)$ [DM91, Chapter 3]. Let $O \subseteq \mu^{-1}(\lambda)$ be an F-stable orbit, then O contains a $\mathbb{F}_q$-point [DM91, Corollary 3.12] which, as we have seen before, has trivial stabilizer. Then, the set of points fixed by F, $O^F \subseteq \mu^{-1}(\lambda)(\mathbb{F}_q)$, is a single $\mathrm{G}_d(\mathbb{F}_q)$-orbit [DM91, Proposition 3.21], that is to say, every F-stable orbit contains a unique $\mathrm{G}_d(\mathbb{F}_q)$-orbit. Therefore

$$\left|\frac{\mu^{-1}(\lambda)(\mathbb{F}_q)}{\mathrm{G}_d(\mathbb{F}_q)}\right| = \left|\left(\frac{\mu^{-1}(\lambda)}{\mathrm{G}_d}\right)^F\right|.$$

Finally, as X_λ is a geometric quotient and $X_\lambda^F = X_\lambda(\mathbb{F}_q)$ (the set of points fixed by the Frobenius automorphism are precisely the $\mathbb{F}_q$-points of the variety [Let18, Lemma 5]) we will have that

$$\left|\left(\frac{\mu^{-1}(\lambda)}{\mathrm{G}_d}\right)^F\right| = |X_\lambda^F| = |X_\lambda(\mathbb{F}_q)|$$

which concludes the proof.

□

Chapter 4

Counting absolutely indecomposable representations

In the previous chapter we saw that if we know that number of $\mathbb{F}_q$-points of X_λ then we can calculate $\mathcal{A}_{Q,d}(\mathbb{F}_q)$. Unfortunately, there is no much more information we can get from X_λ and this is why we fitted X_λ in the one-parameter family $f : \Xi \to L$. This chapter revolves around relating the number of $\mathbb{F}_q$-points of the generic and special fibers. After this, we will see that the $\mathbb{F}_q$-point count of X_0 can be quite simplified as the Frobenius morphism action on its étale cohomology groups satisfies special properties. Finally, at the end of the chapter, we will put all together and give the promised formula for $\mathcal{A}_{Q,d}(\mathbb{F}_q)$.

4.1 Hyperkähler structure on quiver varieties

In this section we work over the field of complex numbers. Our purpose is to introduce the hyperkähler structure associated to the cotangent bundle $T^*\text{Rep}(Q,d) \cong \text{Rep}(\overline{Q},d)$. After this, we will discuss a useful application of this to the problem at hand, which is computing $\mathcal{A}_{Q,d}(\mathbb{F}_q)$. The main references we follow are [Hos18, Section 4.2 and Section 5.1] and [CBVdB04, Section 2.3].

Consider the hermitian form $H : \text{Rep}(Q,d) \times \text{Rep}(Q,d) \to \mathbb{C}$ given by

$$H(X,Y) = \sum_{\alpha \in E} \text{Tr}(X_\alpha \overline{Y}_\alpha^T)$$

which gives us a symplectic form on $\text{Rep}(Q,d)$ and a canonical identification $\text{Rep}(Q,d) \cong \text{Rep}(Q,d)^\vee$. The isomorphism $\mathbb{C}\times\mathbb{C} \cong \mathbb{H}$ is obtained via the map $(z_1,z_2) \mapsto z_1 - jz_2$ so that we have an identification $T^*\text{Rep}(Q,d) \cong \text{Rep}(Q,d)\times\text{Rep}(Q,d) \cong \mathbb{H}^n$ for some $n \in \mathbb{N}$. Right multiplication, by the quaternions i,j,k, on $\mathbb{H}^n$ induces, respectively, the complex structures I,J,K on $T^*\text{Rep}(Q,d) \cong \text{Rep}(\overline{Q},d)$. As a matter of fact

$$\text{I}(X_\alpha)_{\alpha\in\overline{E}} = (iX_\alpha)_{\alpha\in E},\ \text{J}(X_\alpha,X_{\alpha^*})_{\alpha\in E} = (-\overline{X}^T_{\alpha^*},\overline{X}^T_\alpha)_{\alpha\in E} \text{ and } \text{K}(X_\alpha,X_{\alpha^*})_{\alpha\in E} = (-i\overline{X}^T_{\alpha^*}, i\overline{X}^T_\alpha)_{\alpha\in E}$$

and these endow $T^*\mathrm{Rep}(Q,d)$ with a hyperkähler structure.

The hyperkähler metric g is, by definition, the real part of the quaternionic inner product $\langle z,w\rangle = \sum_{l=1}^n z_l\overline{w}_l$ on $\mathbb{H}^n$ where $\overline{w_l}$ stands for the quaternionic conjugate of w_l, that is, the map $a+ib+jc+kd \mapsto a-ib-jc-kd$. So this metric will be given , for the cotangent bundle $T^*\mathrm{Rep}(Q,d)$, by

$$g(X,Y) = \mathrm{Re}\left(\sum_{\alpha\in\overline{E}} \mathrm{Tr}(X_\alpha \overline{Y}_\alpha^T)\right).$$

One can write $\langle -,-\rangle = g - i\omega_{\mathrm{I}} - j\omega_{\mathrm{J}} - k\omega_{\mathrm{K}}$ where $\omega_{\mathrm{I}},\omega_{\mathrm{J}},\omega_{\mathrm{K}}$ are symplectic forms defined as $\omega_{\mathrm{I}} = g(\mathrm{I}-,-)$ and so on. Indeed

$$\omega_{\mathbb{R}}(X,Y) := \omega_{\mathrm{I}}(X,Y) = \mathrm{Im}\left(\sum_{\alpha\in\overline{E}} \mathrm{Tr}(\overline{X}_\alpha^T Y_\alpha)\right)$$

and $\omega_{\mathbb{C}} := \omega_{\mathrm{J}} + i\omega_{\mathrm{K}}$ is the symplectic form in Equation 3.3.

We now observe that the hermitian form H is invariant with respect to the action of the maximal compact subgroup $\mathrm{U}_d := \prod_{v\in V}\mathrm{U}(\alpha_v) < \mathrm{GL}_d$ on $\mathrm{Rep}(Q,d)$. Then the induced U_d-action on $T^*\mathrm{Rep}(Q,d)$ preserves the symplectic forms $\omega_{\mathrm{I}},\omega_{\mathrm{J}}$ and ω_{K} and so there are associated moment maps $\mu_{\mathrm{I}},\mu_{\mathrm{J}},\mu_{\mathrm{K}} : T^*\mathrm{Rep}(Q,d) \to \mathfrak{u}_d^\vee$, where $\mathfrak{u}_d^\vee$ stands for the Lie algebra of U_d, defined by $\mu_{\mathrm{I}}(X)(Y) = -\frac{1}{2}\omega_{\mathrm{I}}(X,YX)$ and so on. Moreover, if we identify $\mathfrak{u}_d \cong \mathfrak{u}_d^\vee$ via the trace pairing we can write

$$\mu_{\mathbb{R}}(X) := \mu_{\mathrm{I}}(X) = \frac{i}{2}\sum_{\alpha\in\overline{E}}[X_\alpha,\overline{X}_\alpha^T]$$

and observe that $\mu_{\mathbb{C}} := \mu_{\mathrm{J}} + i\mu_{\mathrm{K}}$ is the moment map in Equation 3.2. Now that we have briefly reviewed the hyperkähler structure associated to $T^*\mathrm{Rep}(Q,d)$ we can state the main result of this section.

Proposition 4.1. The family $f : \Xi \to L$ is topologically trivial [CBVdB04, Lemma 2.3.3].

Proof. We start by combining the three moment maps described above into the so called *hyperkähler moment map*:

$$\begin{aligned}\mu_{\mathrm{HK}} : \mathrm{Rep}(\overline{Q},d) &\to \mathrm{Im}(\mathbb{H})\otimes_{\mathbb{R}}\mathfrak{u}_d^\vee\\ X &\mapsto i\otimes\mu_{\mathrm{I}}(X) + j\otimes\mu_{\mathrm{J}}(X) + k\otimes\mu_{\mathrm{K}}(X).\end{aligned}$$

Now, observe that μ_{HK} is $\mathbb{H}^*$-equivariant with respect to the left action of $\mathbb{H}^*$ on $\mathrm{Im}(\mathbb{H})$ given by $\beta\cdot\alpha = \beta\alpha\overline{\beta}$. Indeed, $\mu_{\mathrm{HK}}(X\cdot\beta) = \beta\mu_{\mathrm{HK}}(X)\overline{\beta}$ for any $X\in\mathrm{Rep}(\overline{Q},d)$ and $\beta\in\mathbb{H}^*$.

The restriction of the $\mathbb{H}^*$-action to the set $\mathrm{Im}(\mathbb{H})^0 := \mathrm{Im}(\mathbb{H})\setminus\{0\}$ is transitive so for a fixed $\alpha\in\mathrm{Im}(\mathbb{H})^0$, the map $-\cdot\alpha : \mathbb{H}^* \to \mathrm{Im}(\mathbb{H})^0$ admits a continuous section $s : C \to \mathbb{H}^*$ for $C\subset\mathrm{Im}(\mathbb{H})^0$ contractible which contains α. Then, for any coadjoint fixed point $\lambda\in\mathfrak{u}_d^\vee$, there is a local continuous trivialization

$$\mu_{\mathrm{HK}}^{-1}(\alpha\otimes\lambda)\times C \cong \mu_{\mathrm{HK}}^{-1}(C\otimes\lambda)$$
$$(X,c)\mapsto X\cdot s(c)$$

which is U_d-equivariant and therefore there is a homeomorphism

$$(\mu_{\mathrm{HK}}^{-1}(\alpha\otimes\lambda)\times C)/\mathrm{U}_d \cong \mu_{\mathrm{HK}}^{-1}(C\otimes\lambda)/\mathrm{U}_d.$$

We consider $\alpha = i$ and the subset $C = \{i + j\mathbb{C}\} \cong \mathbb{C}$. By construction we have that $\mu_{\mathrm{HK}}^{-1}(\alpha\otimes\lambda) = \mu_{\mathbb{R}}^{-1}(\lambda)\cap\mu_{\mathbb{C}}^{-1}(0)$ and $\mu_{\mathrm{HK}}^{-1}(C\otimes\lambda) = \mu_{\mathbb{R}}^{-1}(\lambda)\cap\mu_{\mathbb{C}}^{-1}(\mathbb{C}\lambda) = \mu_{\mathbb{R}}^{-1}(\lambda)\cap\mu_{\mathbb{C}}^{-1}(L)$ so that the previous discussion gives us a trivialization

$$(\mu_{\mathbb{R}}^{-1}(\lambda)\cap\mu_{\mathbb{C}}^{-1}(0))/\mathrm{U}_d\times\mathbb{C} \cong (\mu_{\mathbb{R}}^{-1}(\lambda)\cap\mu_{\mathbb{C}}^{-1}(L))/\mathrm{U}_d.$$

Finally, we remark that the Kempf-Ness theorem [MFK94, Theorem 8.3] can be generalized to the GIT quotients obtained from linearizations and the moment maps coming from hyperkähler structures. This gives us the homeomorphism

$$X_0\times\mathbb{C}\cong\Xi.$$

We refer the reader to [Kir16, Section 9.10] for a more detailed exposition of this version of the Kempf-Ness theorem. □

4.2 Étale cohomology and point counting over finite fields

Let X be a variety over $\overline{\mathbb{F}}_q$. By an $\mathbb{F}_q$*-structure* on X we mean a variety X' over $\mathbb{F}_q$ such that $X \cong X'\times_{\mathbb{F}_q}\mathrm{Spec}(\overline{\mathbb{F}}_q)$. As pointed out in the proof of Theorem 3.15, an $\overline{\mathbb{F}}_q$-structure comes equipped with a morphism $F : X\to X$, known in the literature as the *geometric Frobenius morphism*, which can be regarded as an extension of the usual Frobenius automorphism and whose fixed locus is the set of $\mathbb{F}_q$-points or *rational* points of the variety X. We refer the reader to [DM91, Chapter 3] for a more detailed exposition of what we have just discussed.

The *Grothendieck trace formula* is a classical result regarding rational point counting in a variety defined over a field of positive characteristic. It reads as follows:

Theorem 4.2. *Let $X = X'\times_{\mathbb{F}_q}\mathrm{Spec}(\overline{\mathbb{F}}_q)$ be an smooth variety over $\overline{\mathbb{F}}_q$ endowed with an $\mathbb{F}_q$-structure and let $F : X\to X$ be the associated geometric Frobenius. Then, for $\mathrm{char}(\mathbb{F}_q)\neq l$,*

$$|X'(\mathbb{F}_{q^n})| = \sum_{i=0}^{2\dim X}(-1)^i\mathrm{Tr}(F^n|H_c^i(X,\mathbb{Q}_l)).$$

Here,

$$F^n : H_c^i(X,\mathbb{Q}_l)\to H_c^i(X,\mathbb{Q}_l)$$

stands for the linear map of $\mathbb{Q}_l$-vector spaces induced by the n-th iteration of the geometric Frobenius and the $H_c^i(X,\mathbb{Q}_l)$ correspond with the compactly supported l-adic cohomology groups of X [Del77, Rapport, Theorem 3.2].

Let us discuss further on the cohomology groups involved in the last theorem. As the usual compactly supported singular cohomology groups for varieties defined over the complex numbers, these cohomology groups have the following properties [Hos18, Section 5.3], [Mil13, Chapter 1]:

- (Finiteness) $H_c^i(X,\mathbb{Q}_l)$ are finite dimensional $\mathbb{Q}_l$-vector spaces.
- (Functoriality) For every proper morphism $\varphi : X \to Y$ there is a map $\varphi^* : H_c^i(Y,\mathbb{Q}_l) \to H_c^i(X,\mathbb{Q}_l)$.
- (Dimensionality) $H_c^i(X,\mathbb{Q}_l) = 0$ for $i > 2\dim X$.
- (Gysin sequences) For any decomposition $X = Z \sqcup U$ with Z closed, there is a long exact sequence
$$\cdots \longrightarrow H_c^i(U,\mathbb{Q}_l) \longrightarrow H_c^i(X,\mathbb{Q}_l) \longrightarrow H_c^i(Z,\mathbb{Q}_l) \longrightarrow H_c^{i+1}(U,\mathbb{Q}_l) \longrightarrow \cdots.$$
- (Leray spectral sequences) For a rank n vector bundle $X \to Y$ we have an isomorphism $H_c^i(X,\mathbb{Q}_l) \cong H_c^{i-2n}(Y,\mathbb{Q}_l) \otimes H_c^2(\mathbb{A}^1,\mathbb{Q}_l)^{\otimes n}$.
- (Poincaré duality) For smooth varieties we have $H^i(X,\mathbb{Q}_l) \cong H_c^{2\dim X - i}(X,\mathbb{Q}_l)$.

In order to conclude something out of the formula in Theorem 4.2 one needs to understand well the eigenvalues of the linear map induced by the geometric Frobenius morphism. Weil's conjectures, proved by Deligne, imply that if X is smooth and proper, then these eigenvalues have all absolute value $q^{i/2}$ for all choice of embedding $\mathbb{Q}_l \hookrightarrow \mathbb{C}$ [Del74, Theorem 1.6]. This prompts the following definition:

Definition 4.3. The variety X is said to be *(cohomologically) pure* if all the eigenvalues of the linear map $F : H_c^i(X,\mathbb{Q}_l) \to H_c^i(X,\mathbb{Q}_l)$ have absolute value $q^{i/2}$.

Now, let us introduce a definition, which combined with the one of purity eases many computations.

Definition 4.4. Let $X = X' \times_{\mathbb{F}_q} \operatorname{Spec}(\overline{\mathbb{F}}_q)$ be a $\overline{\mathbb{F}}_q$–variety together with an $\mathbb{F}_q$-structure. We say that X' is *polynomial count* if there exists a polynomial $p(t) \in \mathbb{Q}[t]$ such that for all $r > 0$, $|X'(\mathbb{F}_{q^r})| = p(q^r)$.

Lemma 4.5. [CBVdB04, Lemma A.1] Assume that X is smooth, pure and polynomial count with counting polynomial being $p(t) \in \mathbb{Q}[t]$. Then:

1. X does not have odd cohomology.
2. $$p(q) = \sum_{i=0}^{\dim X} \dim H_c^{2i}(X,\mathbb{Q}_l) q^i$$
   and in particular $p(t) \in \mathbb{N}[t]$.

Example 4.6. Consider the projective space over $\mathbb{F}_q$. By direct calculation we can see that

$$|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + \ldots + q^n$$

which agrees with the previous lemma as the etale cohomology groups of $\mathbb{P}^n$ are

$$H_c^i(\mathbb{P}^n, \mathbb{Q}_l) = \begin{cases} \mathbb{Q}_l & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

4.3 Purity of the special fiber

The purpose of this section is to show that the variety X_0 is pure. Let us recall, from Section 3.3, that

$$X_0 = \mu^{-1}(0) \cap \mathrm{Rep}(\overline{Q}, d)^{\lambda-ss} /\!\!/_{\chi_\lambda} \mathrm{G}_d$$

and, from Section 2.3, that there is a projective morphism $p : X_0 \to \mathrm{Aff}(X_0) := \mu^{-1}(0) /\!\!/ \mathrm{G}_d$. The idea is to see that there is an induced action of the torus, $\mathbb{G}_m$, on these varieties which gives rise to a Bialynicki-Birula decomposition. This decomposition together with the properties of étale cohomology discussed in the previous section will allow us to conclude on the desired purity. We start with a useful definition introduced in [HRV15].

Definition 4.7. A $\mathbb{G}_m$-action on a smooth quasi-projective variety X is said to be *semi-projective* if $X^{\mathbb{G}_m}$ is projective and for all $x \in X$, $\lim_{t\to 0} t \cdot x$ exists in X .

The torus $\mathbb{G}_m$ acts on $\mathrm{Rep}(\overline{Q}, d)$ by scalar mulitplication and the unique fixed point is the origin. In fact, the limit of every point in $\mathrm{Rep}(\overline{Q}, d)$ under the action of $t \in \mathbb{G}_m$ as $t \to 0$, exists and is equal to the origin so this action is semi-projective. Moreover, the action commutes with the GL_d action on $\mathrm{Rep}(\overline{Q}, d)$ and the algebraic moment map is $\mathbb{G}_m$-equivariant with respect to this action and the weight 2 action of $\mathbb{G}_m$ on the lie algebra $\mathfrak{gl}_d$. Hence, there is an induced action of $\mathbb{G}_m$ on $\mu^{-1}(0)$ and the GIT quotients X_0 and $\mathrm{Aff}(X_0)$ such that $p : X_0 \to \mathrm{Aff}(X_0)$ is $\mathbb{G}_m$-equivariant [Hos18, Section 5.2].

With the previous discussions in mind one can show:

Proposition 4.8. The scaling action $\mathbb{G}_m$ on X_0 is semi-projective [Hos18, Proposition 5.7].

Proposition 4.9. Let X be a smooth quasi-projective variety together with a semiprojective $\mathbb{G}_m$-action, then X is pure [CBVdB04, Proposition A.2].

Proof. Let $X^{\mathbb{G}_m} = \bigcup_{j\in J} X_j$ be the decomposition of the fixed locus into connected components. The hypotheses imply that there is a Bialynicki-Birula decomposition [BB73, Theorem 4.1]:

$$X = \bigsqcup_{j\in J} X_j^+$$

where $X_j^+ = \{x \in X | \lim_{t\to 0} t \cdot x \in X_j\}$ and the limit map $X_j^+ \to X_j$ is a vector bundle. Moreover, since X is quasi-projective, there is a filtration

$$\emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_n = X$$

such that $Y_{i+1} \setminus Y_i$ is equal to one of the X_j^+ [BB76, Theorem 3].

By hypothesis, we have that the smooth varieties X_j are projective and, therefore, they are pure [Del74, Theorem 1.6]. One can see that $H_c^i(\mathbb{A}^1, \mathbb{Q}_l)$ is a one-dimensional vector space for $i = 2$ and zero-dimensional for $i < 2$. Therefore $F^* : H_c^2(\mathbb{A}^1, \mathbb{Q}_l) \to H_c^2(\mathbb{A}^1, \mathbb{Q}_l)$ is the morphism given by multiplication by a scalar λ. By the Grothendieck trace formula we can then conclude that $\lambda = q$. Thus, by the etale cohomology property on Leray spectral sequences in Section 4.2, we can deduce that the X_j^+ are also pure since the eigenvalues of the Frobenius morphism acting on $H_c^i(X_j^+, \mathbb{Q}_l) \cong H_c^{i-2n}(X_j, \mathbb{Q}_l) \otimes H_c^2(\mathbb{A}^1, \mathbb{Q}_l)^{\otimes n}$ have absolute value equal to $q^{(i-2n)/2} q^n = q^{i/2}$.

The last step is to work with the aforementioned filtration and the Gysin sequences introduced in the previous section. At each stage of the filtration we obtain, by the purity of X_j^+, a short exact sequence of vector spaces which splits and gives purity for all Y_j. Indeed, note that $Y_1 = X_{j_1}^+$ and $Y_2 \setminus Y_1 = X_{j_2}^+$ for some $j_1, j_2 \in J$. Then, we have the short exact sequence:

$$0 \longrightarrow H_c^{2i}(X_{j_1}^+, \mathbb{Q}_l) \longrightarrow H_c^{2i}(Y_2, \mathbb{Q}_l) \longrightarrow H_c^{2i}(X_{j_2}^+, \mathbb{Q}_l) \longrightarrow 0.$$

which gives the isomorphism $H_c^{2i}(Y_2, \mathbb{Q}_l) \cong H_c^{2i}(X_{j_1}^+, \mathbb{Q}_l) \oplus H_c^{2i}(X_{j_2}^+, \mathbb{Q}_l)$ and the purity of Y_2. By repeating this process we can conclude that $Y_n = X$ is pure. □

Corollary 4.10. The special fiber, X_0, is pure.

Proof. This follows from a direct application of the last two propositions. □

4.4 Relating the cohomology of the special and generic fibers

In this section, we would like to relate the cohomology groups of the special and generic fibers of the one-parameter family $f : \Xi \to L$ and by doing this, the counts $|X_0(\mathbb{F}_q)|$ and $|X_\lambda(\mathbb{F}_q)|$. To do so, we will need to base change the schemes at hand between the field of complex numbers and various finite fields. However, we are working with GIT quotients and taking invariants and semistable sets does not commute with base changing over arbitrary rings. For us, it will be sufficient to check this commutativity over a special finitely generated $\mathbb{Z}$-algebra.

The $\mathbb{Z}$-algebra we will be working with is $R := \mathbb{Z}[1/N]$ for $N \in \mathbb{Z}$ such that $p \nmid N$. If $\mathcal{G}$ is a reductive group scheme over R acting on a quasi-projective R-variety, X, with respect to the linearisation induced by a character, the formation of the GIT quotient commutes with the base changes previously discussed. More explicitly, we have the following result due to Seshadri [Ses77] and which we will present as in [CBVdB04, Appendix B].

Theorem 4.11. Under the hypothesis of the previous paragraph, there is a non-empty open subscheme $U \subset \mathrm{Spec}(\mathbb{Z}[1/N])$ over which taking GIT semistable sets and GIT quotients commutes with base change. That is, for all points $s : \mathrm{Spec}(k) \to U$, we have

$$X^{ss} \times_{\mathbb{Z}[\frac{1}{N}]} \mathrm{Spec}(k) = (X \times_{\mathbb{Z}[\frac{1}{N}]} \mathrm{Spec}(k))^{ss} \text{ and } (X /\!\!/ \mathcal{G}) \times_{\mathbb{Z}[\frac{1}{N}]} \mathrm{Spec}(k) \cong (X \times_{\mathbb{Z}[\frac{1}{N}]} \mathrm{Spec}(k)) /\!\!/ (\mathcal{G} \times_{\mathbb{Z}[\frac{1}{N}]} \mathrm{Spec}(k)).$$

We now specialize all this to the quiver varieties setting. The framework is the following: we can define the affine space $\mathrm{Rep}(\overline{Q}, d)$, the group GL_d, the moment map μ and the corresponding GIT quotients, X_λ and X_0, over the finitely generated $\mathbb{Z}$-algebra $\mathbb{Z}[1/N]$. It is worth mentioning that to consider these GIT quotients over this $\mathbb{Z}$-algebra is possible thanks to Seshadri's generalization of GIT to arbitrary base rings [Ses77, Section 1.2].

On the other hand, observe that an open subset $U \subset \mathrm{Spec}(\mathbb{Z}[1/N])$ is of the form $U = \mathrm{Spec}(\mathbb{Z}[1/M])$ for some $N|M$. Since $p \nmid N$ we can base change to $\mathbb{F}_p$ and all its field extensions [Hos18, Section 5.6]. By considering the generic point of U, we see that we can base change to $\mathbb{Q}$ and all its fields extensions, such as $\mathbb{C}$. In brief, we have the following base changes

$$\begin{array}{ccccccc}
X_{\overline{\mathbb{F}}_p} & \longrightarrow & X_{\mathbb{F}_{p^r}} & \longrightarrow & X & \longleftarrow & X_{\mathbb{C}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(\overline{\mathbb{F}}_p) & \longrightarrow & \mathrm{Spec}(\mathbb{F}_{p^r}) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}]) & \longleftarrow & \mathrm{Spec}(\mathbb{C})
\end{array}$$

and for N large enough, these base changes commute with GIT quotients and semistable sets by the previous theorem.

Proposition 4.12. For a finite field $\mathbb{F}_q$ of sufficiently large characteristic p, we have

$$|X_0(\mathbb{F}_q)| = |X_\lambda(\mathbb{F}_q)|.$$

Proof. The strategy of Crawley-Boevey and Van der Bergh is to show that the etale cohomology groups of the special and generic fibers coincide when the characteristic of the field is taken to be large enough. We now briefly explain the proof of this fact.

As in Lemma 3.12, we assume that Ξ and L are defined over $\mathbb{Z}$ so the first step is to check that $R^i f_!(\mathbb{Q}_l)$ is constant over an open subset of the base $\mathrm{Spec}(\mathbb{Z})$ [CBVdB04, Proposition 2.3.1]. To see this, one uses Deligne's generic base change theorem for direct images [Del77, Th. Finitude, Theorem 1.9], the topological triviality of the family $f : \Xi \to L$ over $\mathbb{C}$ that we showed in Proposition 4.1 and the comparison theorem between etale and singular cohomologies [BBD82, Chapter 6 Section 1.2].

Using the previous result and the fact that $R^i f_!$ commutes with base change one can see, for $p \gg 0$ that $R^i f_{0!}\mathbb{Q}_l \cong R^i f_{\lambda !}\mathbb{Q}_l$ on $\mathrm{Spec}(\mathbb{F}_p)$ where f_0 and f_λ are the restrictions of the morphism f to the special and generic fibers respectively. Now, one can see that $R^i f_{0!}(\mathbb{Q}_l) \cong H^i_c(X_{0,\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$

and $R^i f_{\lambda !}(\mathbb{Q}_l) \cong H^i_c(X_{\lambda,\overline{\mathbb{F}}_q},\mathbb{Q}_l)$ so that $H^i_c(X_{0,\overline{\mathbb{F}}_q},\mathbb{Q}_l) \cong H^i_c(X_{\lambda,\overline{\mathbb{F}}_q},\mathbb{Q}_l)$. One of the key points is that this isomorphism is compatible with the action of the Frobenius morphism which means that the rational point count that we will obtain by the Grothendieck trace formula, in Theorem 4.2, will be the same.

There is however, another proof of this fact, due to Nakajima [CBVdB04, Appendix by Hiraku Nakajima], involving the Bialynicki-Birula decomposition of both the total space of the family $f : \Xi \to L$ and the special fiber, X_0, with respect to an action of the torus, $\mathbb{G}_m$, on Ξ which we now review. This action, introduced by Nakajima in [Nak94, Section 5], is such that f is $\mathbb{G}_m$-equivariant with respect to the weight one action of the torus on the affine line L and such that for every $\xi \in \Xi$, $\lim_{t\to 0} t\cdot\xi$ exists.

The first observation is that X_{t_1} is isomorphic to X_{t_2} for all non-zero $t_1, t_2 \in L \cong \mathbb{A}^1_k$. Thus, for us will be enough to show that $|X_0(\mathbb{F}_q)| = |X_1(\mathbb{F}_q)|$. Let $\Xi^{\mathbb{G}_m} = \bigcup_{j\in J} Y_j$ be the decomposition of the fixed locus of the $\mathbb{G}_m$-action into connected components. Clearly, each of the Y_j is a smooth projective variety contained in X_0 as the map f is $\mathbb{G}_m$-equivariant with respect to the weight one action of the torus on L. The Bialynicki-Birula decomposition of Ξ [BB73, Theorem 4.1] is then given by: $\Xi = \bigsqcup_{j\in J} \Xi_j$ where $\Xi_j = \{\xi \in \Xi | \lim_{t\to 0} t\cdot\xi \in Y_j\}$ and the limit map $\Xi_j \to Y_j$ is a vector bundle. Thus we have that

$$|\Xi(\mathbb{F}_q)| = \sum_{j\in J} |\Xi_j(\mathbb{F}_q)| = \sum_{j\in J} q^{n_j} |Y_j(\mathbb{F}_q)|$$

where n_j stands for the rank of the aforementioned vector bundles.

One can also consider the Bialynicki-Birula decomposition of the special fiber, $X_0 = \bigsqcup_{j\in J} X_{0,j}$ where the $X_{0,j}$ are defined as usual and the limit maps $X_{0,j} \to Y_j$ are as well vector bundles. Now, the key observation here is that the tangent space of Ξ, at a point in $\Xi^{\mathbb{G}_m}$, decomposes as the sum of the tangent space of X_0, which corresponds to the fiber direction, plus a one-dimensional space corresponding to the base direction. Thus, $X_{0,j} \to Y_j$ is a vector bundle of rank $n_j - 1$. It follows that

$$|X_0(\mathbb{F}_q)| = \sum_{j\in J} |X_{0,j}(\mathbb{F}_q)| = \sum_{j\in J} q^{n_j - 1} |Y_j(\mathbb{F}_q)| = \frac{1}{q} |\Xi(\mathbb{F}_q)|.$$

Finally, note that

$$|\Xi(\mathbb{F}_q)| = \sum_{\lambda\in\mathbb{F}_q} |X_\lambda(\mathbb{F}_q)| = (q-1)|X_1(\mathbb{F}_q)| + |X_0(\mathbb{F}_q)|.$$

By combining the previous two equalities we obtain the desired result. □

4.5 Kac's conjectures and count of absolutely indecomposable representations

Before stating the main result of this memoir, we recall a result due to Kac which was one of the motivations for the work of Crawley-Boevey and Van der Bergh [CBVdB04].

Theorem 4.13. The number of absolutely indecomposable quiver representations over $\mathbb{F}_q$, $\mathcal{A}_{Q,d}(\mathbb{F}_q)$, does not depend on the orientation of Q. Moreover, $\mathcal{A}_{Q,d}(\mathbb{F}_q)$ is a polynomial in q with integer coefficients [Kac83, Chapter 1, Section 13, Lemma].

The following theorem, besides providing us with a formula to count the absolutely indecomposable representations we are interested in, also shows a conjecture by Kac regarding the positivity of the coefficients of the polynomial $\mathcal{A}_{Q,d}(\mathbb{F}_q)$ under some assumptions [Kac83, Chapter 1, Section 15, Conjecture 2].

Theorem 4.14. Let Q be a quiver without loops and $d \in \mathbb{N}^{|V|}$ an indivisible dimension vector. For a generic stability parameter λ with respect to d and for a finite field of sufficiently large characteristic, we have

$$\mathcal{A}_{Q,d}(\mathbb{F}_q) = \sum_{i=0}^{e} \dim H^{2e-2i}(X_0, \mathbb{C}) q^i$$

where $e = \frac{1}{2}\dim X_0$ and X_0 is the moduli space of λ-stable representations in $\mathrm{Rep}(\Pi^0, d)$.

Proof. Let $\mathbb{F}_q$ be a finite field of characteristic p sufficiently large such that Lemma 3.12 and Theorem 4.11 hold. By Theorem 3.15 and Theorem 4.13 we deduce that the $\mathbb{F}_q$-variety X_λ is polynomial count with counting polynomial given by the equality

$$\mathcal{A}_{Q,d}(\mathbb{F}_q) = q^{-e}|X_\lambda(\mathbb{F}_q)|.$$

Since p is sufficiently large, by Proposition 4.12, this point count coincides with the one of the special fiber X_0. Recall, by Corollary 4.10, that X_0 is pure and since smooth, by Lemma 4.5,

$$\mathcal{A}_{Q,d}(\mathbb{F}_q) = \sum_{i=0}^{2e} \dim H_c^{2i}(X_0, \mathbb{Q}_l) q^{i-e}$$

where $e = \frac{1}{2}\dim X_0$ and $l \neq p$ prime.

We can consider the family $f : \Xi \to L$ over $\mathbb{Z}[1/N]$, with N sufficiently large such that $p \nmid N$, as in the past section. Thus, we can base change X_0 from $\overline{\mathbb{F}}_q$ to $\mathbb{C}$ and obtain a complex variety $X_{0,\mathbb{C}}$. Is important to observe that this variety, by the GIT commutativity stated in Theorem 4.11, is the special fiber of the one-parameter family $f : \Xi \to L$ when the ground field is taken to be the field of complex numbers. Thus, by the smooth base change theorems and the comparison theorems [Mil80, Chapter VI, Corollary 4.2], [BBD82, Chapter 6 Section 1.2] we obtain

$$\mathcal{A}_{Q,d}(\mathbb{F}_q) = \sum_{i=0}^{2e} \dim H_c^{2i}(X_{0,\mathbb{C}}, \mathbb{C}) q^{i-e} = \sum_{i=0}^{e} \dim H_c^{2i+2e}(X_{0,\mathbb{C}}, \mathbb{C}) q^i.$$

Here, the right-most equality is obtained from the fact that $\mathcal{A}_{Q,d}(\mathbb{F}_q)$ must be a polynomial in q and the cohomology groups are the singular cohomology groups of the complex variety $X_{0,\mathbb{C}}$. From Poincaré duality, for the smooth variety $X_{0,\mathbb{C}}$, we then have

$$\mathcal{A}_{Q,d}(\mathbb{F}_q) = \sum_{i=0}^{e} \dim H^{2e-2i}(X_{0,\mathbb{C}}, \mathbb{C}) q^i$$

which is the desired result.

□

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