A gentle introduction to the Hitchin Fibration

$X :=$ connected Riemann surface of genus $g > 1$.

Goal: My goal is to introduce the Moduli space of Higgs bundles over X of fixed rank r and degree d, $\mathcal{M}(r,d)$ and show how can it be better understood through the so-called Hitchin Fibration.

1. Rank 1 Higgs Bundles
2. Higgs Bundles: Definitions and main properties
3. The Hitchin Fibration
4. Spectral curves / covers
5. The Beauville-Narasimhan-Ramanan correspondence

1. Rank 1 - Higgs Bundles

The exact sequence of sheaves

$$\boxed{0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0}$$

induces the following exact sequence of abelian groups

$$0 \to \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z}_X)} \cong \mathbb{C}^g/\mathbb{Z}^{2g} \to \underbrace{H^1(X,\mathcal{O}_X^*)}_{\text{Picard group}} \xrightarrow{\deg} H^2(X,\mathbb{Z}_X) \cong \mathbb{Z} \to 0$$

Moduli space of stable vectorbundles of degree 0 and rank 1.

$\Rightarrow$ $Jac^0(X) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$ (g-holed torus) which is a complex Lie group.

$\Rightarrow$ $T^*Jac^0(X) \cong Jac^0(X) \times H^1(X,\mathcal{O}_X)^* \cong Jac^0(X) \times H^0(X, \underbrace{K_X}_{})$

Serre Duality: $H^1(X,F) \cong H^0(X, F^* \otimes K_X)^*$

K_X: Canonical Bundle, Sheaf of holomorphic differential 1-forms

$\forall$ For any line bundle $L \to X$, $\mathcal{O}_X \cong L^* \otimes L \cong End(L)$

$\Rightarrow$ $T^*Jac^0(X) \cong Jac^0(X) \times H^0(X, End(L) \otimes K_X)$ $\rightsquigarrow$ Every point $(L,\Phi) \in T^*Jac^0(X)$ corresponds to a rank 1 Higgs bundle + degree 0

2. Higgs Bundles: Definitions and main properties

Definition → • A Higgs bundle over X is a pair (E,Φ) where $p: E \to X$ is a holomorphic vector bundle over X and Φ is a differential 1-form with values in $End(E)$, i.e. $\Phi \in H^0(X, End(E) \otimes K_X)$. Φ is known as the Higgs field.

[Riemann Roch Theorem: $h^0(X,E) - h^1(X,E) = \deg(E) + rk(E)(1-g)$

Riemann-Hurwitz: $K_{X_a} \cong \pi_a^* K_X \otimes \mathcal{O}(\Delta)$]

- A Higgs Bundle morphism is a map $f : (E_1, \phi_1) \to (E_2, \phi_2)$ such that

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "\phi_1"'] \arrow[dr, phantom, "\#"] & E_2 \arrow[d, "\phi_2"] \\ E_1 \otimes K_M \arrow[r, "f \otimes id_{K_M}"'] & E_2 \otimes K_M \end{tikzcd}$$

In moduli theory, in order to get "good" moduli spaces one usually imposes a stability condition to the objects one is intending to classify:

Definition → (E, ϕ) is said (semi)-stable if for all ϕ-invariant proper vector bundle $F \subsetneq E$,

$$\boxed{\mu(F) := \frac{\deg(F)}{rk(F)} \ (\leq) < \ \frac{\deg(E)}{rk(E)} \in \mathbb{Q}}$$

Some properties about stability:
$$\left\{ \begin{array}{l} \bullet\ (E, f^{-1}\phi f) \ \forall \text{ automorphism } f : E \to E \\ \bullet\ (E, \lambda\phi) \ \forall \lambda \in \mathbb{C}^* \\ \bullet\ (E, \tilde{\phi}) \text{ for } \tilde{\phi} \text{ "close to" } \phi \end{array} \right.$$
are stable if (E, ϕ) is.

Theorem There exists a smooth complex variety of dimension $2(r^2(g-1)+1)$, $\mathcal{M}(r,d)$, whose points are in 1-1 correspondence with isomorphism classes of stable Higgs bundles of fixed degree d and rank r. We call this, the moduli space of stable Higgs bundles over X.

Remarks.
- Let $\mathcal{U}(r,d)$ denote the moduli space of vector bundles over X => $\boxed{T^*\mathcal{U}(r,d) \subseteq \mathcal{M}(r,d)}$ (r,d) coprime
- (Non-abelian Hodge correspondence)

$$\mathcal{M}(r,0) \overset{\text{Harmonic Bundles}}{\cong} \mathcal{M}_{dR} \overset{\text{Riemann-Hilbert correspondence}}{\cong} \{\rho : \pi_1(X) \to GL(r,\mathbb{C})\}^{irred} / GL(r,\mathbb{C})$$

$\mathcal{M}_{dR}$ ↓ Moduli space of flat bundles of rank r over X

"Character variety"

Example → $K_X^{1/2} :=$ Square root for the canonical bundle K_X (Also called "Spin structure") (2^{2g} of them)

$\omega \in H^0(X, K_X^2)$, $E = K_X^{1/2} \oplus K_X^{-1/2}$, $\phi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix}$.

Note that for all $\alpha \in H^0(X, K_X^{1/2})$, $\beta \in H^0(X, K_X^{-1/2})$,

$$\begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \omega \otimes \beta \\ \alpha \end{pmatrix} \in H^0(X, \underbrace{K_X^{3/2} \oplus K_X^{1/2}}_{\cong (K_X^{1/2} \oplus K_X^{-1/2}) \otimes K_X})$$

=> (E, ϕ) is a rank 2 Higgs bundle. As a matter of fact, one can show this to be a point in $\mathcal{M}(2,0)$. Indeed, $(E,\phi) \in \mathcal{M}(2,0)$.

3. The Hitchin Fibration

Some computations in local coordinates ...

$(E,\phi) \in \mathcal{M}(r,d)$ rank r Higgs bundle.
$U_j \subseteq X$ trivializing open set

$\Rightarrow$ $\boxed{\phi|_{U_j} = A_j \otimes dz_j = \begin{pmatrix} f_{11}(z_j) & \cdots & f_{1r}(z_j) \\ \vdots & & \vdots \\ f_{r1}(z_j) & \cdots & f_{rr}(z_j) \end{pmatrix} \otimes dz_j}$ Holomorphic map

$\Rightarrow$ In two different trivializations the following relation should hold:

$$\boxed{A_i \otimes dz_i = Ad(g_{ij}) A_j \otimes h_{ij}\, dz_j} \quad \text{on } U_{ij}$$

where $Ad(g_{ij})$ and h_{ij} are the corresponding transition functions of the vector bundles $End(E)$ and K_X respectively.

As the Higgs field takes this matricial form it is natural to ask whether one can define analogues of trace, det ?...

$$\boxed{Tr(\phi|_{U_j}) = \sum_{k=1}^{r} f_{kk} \otimes dz_j \Rightarrow Tr(\phi|_{U_i}) = \sum_{k=1}^{r} f_{kk} \otimes h_{ij}\, dz_j = h_{ij}\, Tr(\phi|_{U_j})}$$

$\Rightarrow$ $\{Tr(\phi|_{U_j})\}$ glue well and give rise to a section of K_X which we call the trace of the Higgs bundle and denote by $Tr(\phi) \in H^0(X, K_X)$

Theorem Let $p: \mathfrak{gl}(r,\mathbb{C}) \to \mathbb{C}$ an Ad-invariant homogeneous polynomial with $\deg(p)=n$. Then p induces a well-defined map:

$$\begin{cases} f_p: \mathcal{M}(r,d) \longrightarrow H^0(X, K_X^n) \\ \quad (E,\phi) \longmapsto p(\phi) \end{cases}$$

where the section $p(\phi)$ is defined locally by $p(\phi)|_{U_j} = p(A_j) \otimes dz_j^n$.

Proof:
$p(\phi)|_{U_i} = p(A_i) \otimes dz_i^n$
$= p(Ad(g_{ij})A_j) \otimes (h_{ij}\, dz_j)^n$
$= h_{ij}^{\otimes n}\, p(\phi)|_{U_j}$

Examples: Trace, determinant and the other coefficients of the characteristic polynomial of a matrix which can be written in terms of $Tr(\phi), \dots, Tr(\phi^r)$ (1st fundamental theorem of invariant theory)

3. The Hitchin Fibration

$$\begin{cases} h: \mathcal{M}(r,d) \longrightarrow \overbrace{\mathcal{A}(r) := \bigoplus_{n=1}^{r} H^0(X, K_X^n)}^{\text{Hitchin Base}} \\ \quad (E,\phi) \longmapsto (p_1(\phi), \dots, p_r(\phi)) \end{cases}$$

$p_i(\phi)$:= section associated to the ith coefficient of the characteristic polynomial of a matrix.

Remarks: • h is holomorphic + proper
• $\dim \mathcal{A}(r) = \dfrac{\dim \mathcal{M}(r,d)}{2}$

Moreover,
In fact, h is surjective. In what remains, we attempt to show this and describe the fibers of h.

4. Spectral Covers

We make precise what would be the characteristic polynomial of a Higgs bundle and link the geometry of its zero locus to our problem at hand.

Consider the commutative diagram:

$$\begin{tikzcd}
K_X \arrow[dr, dashed, "\eta"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& \pi^* K_X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\#"] & K_X \arrow[d, "\pi"] \\
& K_X \arrow[r, "\pi"'] & X
\end{tikzcd}$$

$\Rightarrow$ For $(a_1, \dots, a_r) \in \overset{a}{\mathcal{A}}(r)$ we consider the section

$\eta^r + \pi^* a_1 \otimes \eta^{r-1} + \dots + \pi^* a_{r-1} \otimes \eta + \pi^* a_r \in H^0(K_X, \pi^* K_X^r)$

- Taking $a_j = p_j(\phi)$ we recover a section of $\pi^* K_X^r$ which we call the characteristic polynomial of the Higgs bundle (E, ϕ).

Remark:
- Two Higgs bundles have the same image under the Hitchin fibration $\to h$ iff its characteristic polynomials coincide.
- $(F, \phi|_F)$, ϕ-invariant Higgs sub-bundle of $(E, \phi) \Rightarrow$ the characteristic polynomial of (E, ϕ). $(F, \phi|_F) \hookrightarrow (E, \phi) \Rightarrow$ Its characteristic polynomial divides the characteristic polynomial of (E, ϕ).

The zero locus

$$\boxed{K_X \supset X_a = \{ Q \in K_M \mid (\eta^r + \pi^* a_1 \otimes \eta^{r-1} + \dots + \pi^* a_r)(Q) = 0 \}} \,!$$

is known as spectral curve. It not always defines a Riemann surface but...

Proposition: There is an open dense subset of $\mathcal{A}(r)$ for which X_a will have structure of compact Riemann surface.

$\Rightarrow$ $\pi_a : X_a \to X$ will be an r-sheeted branched covering of compact Riemann surface which we called spectral cover.

Consider the formal derivative of the equation defining X_a which is the section

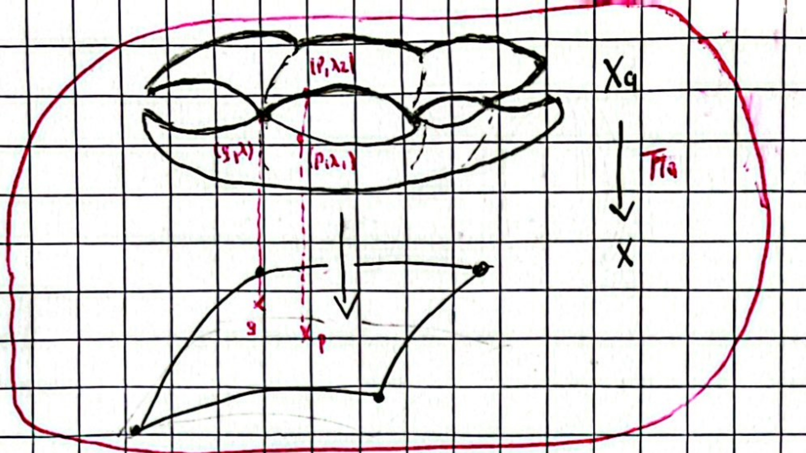
$$\boxed{D := r\eta^{r-1} + (r-1)\pi_a^* a_1 \otimes \eta^{r-2} + \dots + \pi_a^* a_{r-1} \in H^0(K_X, \pi_a^* K_X^{r-1})}$$

We can think of the points in X_a as pairs $Q := (p, \lambda) \in X_a$, $p \in X$, $\lambda \in \mathbb{C}$ is an eigenvalue of the linear map $\phi_p : E_p \to E_p \otimes K_{X,p}$. $\Rightarrow$ Q is a repeated eigenvalue iff $D(Q) = 0$!!!

There is more...

Branching order or multiplicity of the spectral cover at Q.

Theorem: D vanishes at $Q \in X_a$ with multiplicity $\mathrm{mult}_Q(\pi_a) - 1$.
Equivalently, the multiplicity of λ as eigenvalue of ϕ_p is $\mathrm{mult}_Q(\pi_a)$.



Corollaries This theorem has some interesting corollaries:

$$\underbrace{\mathcal{O}(\Delta)}_{\substack{\text{Line bundle}\\\text{associated to the}\\\text{ramification divisor}\\\text{of the spectral cover}}} \cong \pi_a^* K_X^{r-1} \underset{\text{Riemann-Hurwitz}}{\Longrightarrow} K_{X_a} \cong \pi_a^* K_X^{r} \underset{\text{Riemann-Roch}}{\Longrightarrow} g_a = r^2(g-1)+1$$

$$\boxed{\Delta = \sum_{Q\in X_a} (\mathrm{mult}_Q(\pi_a)-1)\cdot Q} \quad (1)$$

5. The Beauville - Narasimhan - Ramanan correspondence

Theorem Let $(a_1,\dots,a_r) = a \in \mathcal{B}(r)$ such that X_a is a Riemann surface. Then,

$$\boxed{\mathrm{Jac}^0(X_a) \overset{1:1}{\longleftrightarrow} h^{-1}(a) \subset \mathcal{M}(r,d).}$$

This implies, in particular, that $h : \mathcal{M}(r,d) \to \mathcal{B}(r)$ is surjective. Hurdiday!

__Idea of the proof:__ Pick $L \to X_a$ line bundle such that $\deg(L) = d + (r-r^2)(1-g)$. Since π_a is an r-sheeted branched covering, $\pi_{a*}L$ is a vector bundle of rank r. Tensorization by the canonical section induces η

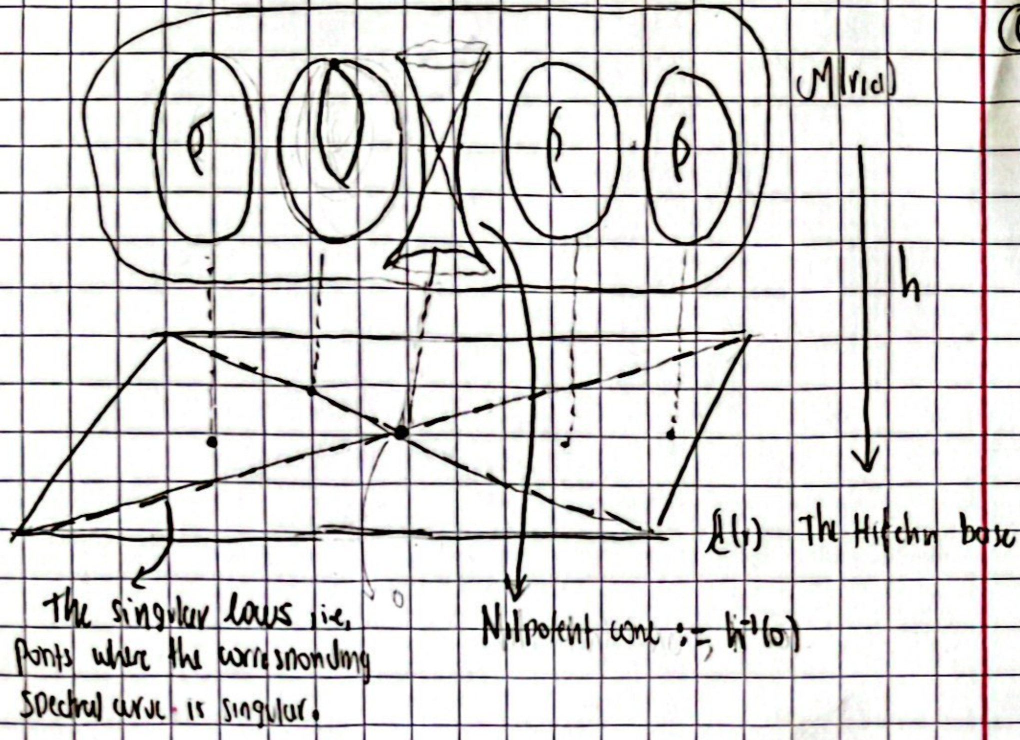
$$H^0(U,\pi_{a*}L) \cong H^0(\pi_a^{-1}(U),L) \xrightarrow{\otimes\eta} H^0(\pi_a^{-1}(U), L\otimes\pi_a^* K_X) \cong H^0(U, \pi_{a*}L\otimes K_X)$$

and therefore defines a Higgs field $\boxed{\pi_{a*}\eta : \pi_{a*}L \longrightarrow \pi_{a*}L \otimes K_X.}$ $\Rightarrow (\pi_{a*}L, \pi_{a*}\eta) \in \mathcal{M}(r,d)$.

Now we go in the other direction. Let $(E,\Phi) \in \mathcal{M}(r,d)$ with spectral curve given by X_a. Recall that the points of the spectral curve are pairs (p,λ), $p\in X$, $\lambda\in\mathbb{C}$ is an eigenvalue $\Phi_p : E_p \to E_p \otimes K_{X,p}$. By attaching to each point in the spectral curve its corresponding eigenspace, we obtain a line bundle $L' \hookrightarrow \pi_a^* E$. The line bundle associated to (E,Φ) will be then $L = L' \otimes \mathcal{O}(\Delta)$.

__Example:__ For $L = \pi_a^* K_X^{1/2}$, the corresponding Higgs bundle will be

$$\pi_{a*} L \cong K_X^{1/2} \oplus K_X^{-1/2}$$


M(r,d)
h
B(r) The Hitchin base
The singular locus, i.e, points where the corresponding spectral curve is singular.
Nilpotent cone := h^{-1}(0)