Part 1 : Character Table of $Gl_2(\mathbb{F}_q)$ Ref: Lang- Algebra

1.1 Conjugacy classes of $Gl_2(\mathbb{F}_q)$

Let $A \in Gl_2(\mathbb{F}_q) \Rightarrow \det(A - \lambda Id) = (\lambda - \alpha_1)(\lambda - \alpha_2)$ with $\alpha_1, \alpha_2 \in \mathbb{F}_q^{\times}$

or

$\det(A - \lambda Id) = \lambda^2 - (\varepsilon + \varepsilon^q)\lambda + \varepsilon\varepsilon^q$ with $\varepsilon \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$

$\Rightarrow$ A is conjugate to one of the following 4 matrices:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} 0 & -\varepsilon\varepsilon^q \\ 1 & \varepsilon+\varepsilon^q \end{pmatrix} \quad \begin{cases} \alpha \neq \beta \in \mathbb{F}_q^{\times} \\ \varepsilon \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \end{cases}$$

(i) (ii) (iii) (iv)

Now, we calculate the number of conjugacy classes of each type:

(i) There are $\underline{q-1}$ different classes, one for each $\alpha \in \mathbb{F}_q^{\times}$.

(ii) There are $\underline{q-1}$ different classes, one for each $\alpha \in \mathbb{F}_q^{\times}$.

(iii) There are $\frac{(q-1)(q-2)}{2}$ since $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and $\begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$ are in the same class.

(iv) A matrix of this type is $Gl_2(\mathbb{F}_{q^2})$ conjugate to $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^q \end{pmatrix} \Rightarrow$ There are $\frac{q^2-q}{2}$ different classes.
Again, we need to take into account that $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^q \end{pmatrix}$ and $\begin{pmatrix} \varepsilon^q & 0 \\ 0 & \varepsilon \end{pmatrix}$ are in the same conjugate class.

We summarize our results in the following table:

Type of conjugacy class	# of different conjugacy classes	# elements in each conjugacy class
$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	$q-1$	1
$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$	$q-1$	q^2-1
$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$	$\frac{(q-1)(q-2)}{2}$	$q(q+1)$
$\begin{pmatrix} 0 & -\varepsilon\varepsilon^q \\ 1 & \varepsilon+\varepsilon^q \end{pmatrix}$	$\frac{q(q-1)}{2}$	q^2-q

(Rows 1–3 of the last column: Calculated by brute force using Orbit Stabilizer thm.)

1.2 Type I characters of $Gl_2(\mathbb{F}_q)$

$\mu: \mathbb{F}_q^{\times} \to \mathbb{C}^{\times}$ group homomorphism $\Rightarrow \mu \circ \det: Gl_2(\mathbb{F}_q) \to \mathbb{C}^{\times}$ is a 1-dimensional character and hence irreducible. Its values in the conjugacy classes described above are

χ	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \alpha \neq \beta$	$\begin{pmatrix} 0 & -\varepsilon\varepsilon^q \\ 1 & \varepsilon+\varepsilon^q \end{pmatrix}$
$\mu \circ \det$	$\mu(\alpha)^2$	$\mu(\alpha)^2$	$\mu(\alpha)\mu(\beta)$	$\mu(\varepsilon\varepsilon^q)$

¿ How many different characters are of this type? As many as group homomorphisms $\mu: \mathbb{F}_q^{\times} \to \mathbb{C}^{\times}$.
$\mathbb{F}_q^{\times} \cong \mathbb{Z}/(q-1)\mathbb{Z} \Rightarrow \mu$ is determined by the image of the generator of this group, $\xi = e^{2\pi i/q-1}$.

Note that $\mu(\xi)^{q-1} = \mu(\xi^{q-1}) = 1 \Rightarrow \mu(\xi)$ must be a $q-1$-root of unity.

$\Rightarrow$ We have $q-1$ homomorphisms $\mu: \mathbb{F}_q^{\times} \to \mathbb{C}^{\times}$, one for each $q-1$ root of unity.

1.3 Type II characters of $GL_2(\mathbb{F}_q)$

$U :=$ Unipotent elements, $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$

$T :=$ Diagonal subgroup of $GL_2(\mathbb{F}_q)$, $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

$B := UT = TU \longrightarrow$ Subgroup of upper Δ'r matrices, also known as the Borel subgroup.

Observe that $B/U = T \Rightarrow$ A character of T can be viewed as a character on B via B/U.

Consider $\psi_\mu := \operatorname{res}_T(\mu \circ \det) \Rightarrow \psi_\mu$ can be viewed as a character on B.

$$\psi_\mu \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix} = \mu(\alpha)\mu(\beta).$$

$\Rightarrow$ We can induce a character on G, $\psi_\mu^G = \operatorname{ind}_B^G(\psi_\mu)$.

However, ψ_μ^G is not simple since it contains a copy of $\mu \circ \det$. Indeed by Frobenius reciprocity theorem:

$$\langle \operatorname{ind}_B^G \psi_\mu, \mu\circ\det\rangle_G = \langle \psi_\mu, \mu\circ\det\rangle_B$$
$$= \frac{1}{|B|}\sum_{M\in B} |\mu\circ\det(M)|^2 = 1$$

$\Rightarrow$ The characters we are interested in, the Type II characters, are $\psi_\mu^G - \mu\circ\det$. Its value in the conjugacy classes are:

χ	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \alpha\neq\beta$	$\begin{pmatrix} 0 & -\varepsilon\varepsilon^q \\ 1 & \varepsilon+\varepsilon^q \end{pmatrix}$
$\psi_\mu^G - \mu\circ\det$	$q\,\mu(\alpha)^2$	0	$\mu(\alpha)\mu(\beta)$	$-\mu\circ(\varepsilon\varepsilon^q)$

¿ How did we calculate these values? We used the induced character formula $\operatorname{ind}_B^G(\psi_\mu)(A) = \frac{1}{|B|}\sum_{\substack{M\in GL_2(\mathbb{F}_q) \\ M^{-1}AM\in B}} \psi_\mu(M^{-1}AM)$ and counted how many matrices $M \in GL_2(\mathbb{F}_q)$ satisfy $M^{-1}AM \in B$ for A in each one of the conjugacy classes. $= \frac{1}{|B|}\sum_{\substack{M\in GL_2(\mathbb{F}_q) \\ M^{-1}AM\in B}} \mu\circ\det(A)$

¿ Are characters of type II irreducible? By orthogonality of characters we can see that
$$\langle \chi,\chi\rangle = \frac{1}{|GL_2(\mathbb{F}_q)|}\sum_{M\in GL_2(\mathbb{F}_q)} \chi(M)\overline{\chi}(M) = 1.$$

¿ How many characters are of this type? These depend only of $\mu: \mathbb{F}_q^{\times} \to \mathbb{C}^{\times} \Rightarrow$ There are $q-1$ of them, by the same analysis we did for type I.

1.4 Type III characters of $GL_2(\mathbb{F}_q)$

$\psi: T \to \mathbb{C}^{\times}$ group homomorphism.

For $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, note that $\underbrace{w}_{} \underset{A}{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}} \underset{w}{w^{-1}} = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} = A^w \Rightarrow$ Conjugation by w is an order 2 automorphism of T.

Define $([w]\psi)(A) := \psi(A^w) \rightsquigarrow$ character conjugado $\Rightarrow$ Observe that $([w]$ [illegible]

¿How many characters are of type III?

These characters depend on the morphism $\psi : T \to \mathbb{C}^{\times}$ choosen (3.5)

We must recall as well that ψ ± prodct which means that if $\psi\begin{pmatrix}\alpha & 0 \\ 0 & \beta\end{pmatrix} = \psi_1(\alpha)\psi_2(\beta) \Rightarrow \psi_1 \neq \psi_2$.

We have $q-1$ possibilities to send the generator of the $q-1$-roots of unit and and this determine ψ_1.

For ψ_2 we have $q-2$ possibilities to send this generator.

But observe that $\psi^G([w]M) = \psi^G(M)$ in each conjugacy class,

then we have $\frac{(q-1)(q-2)}{2}$ different characters

The only characters invariant under $[w]$ are those of the type product. The others can be written as

$$\psi'\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} = \psi_1(\alpha)\,\psi_2(\beta)$$

with $\psi_1, \psi_2 : \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $\psi_1 \neq \psi_2$.

Given that $B/_U \cong T \Rightarrow \psi$ can be seen as a character over B

$\Rightarrow \quad \psi^G = \mathrm{ind}_B^G([w]\psi)$ with ψ such that $\psi \neq [w]\psi$.

This characters will be called of Type III.

Its value in the conjugacy classes of $Gl_2(\mathbb{F}_q)$ is:

χ	$\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$	$\begin{pmatrix}0 & -\varepsilon\varepsilon^q\\ 1 & \varepsilon+\varepsilon^q\end{pmatrix}$
ψ^G $\psi\neq[w]\psi$	$q+1\ \psi(\alpha)$	$\psi(\alpha)$	$\psi(\alpha)+\psi(\alpha^w)$	0

¿ How did we calculate this values? Again we use the induced character formula

$$\mathrm{ind}_B^G(\psi^G)(A) = \sum_{\substack{M\in Gl_2(\mathbb{F}_q)\\ M^{-1}AM\in B}} \psi(wM^{-1}AMw)$$

and calculate according to the conjugacy class.

¿ Are characters of type III irreducible? Note that $\sum_{M\in Gl_2(\mathbb{F}_q)} |\psi'^G(M)|^2$

$$= \sum_{\alpha\in\mathbb{F}_q^\times} |(q+1)\psi(\alpha)|^2 + \sum_{\alpha\in\mathbb{F}_q^\times} (q^2-1)|\psi(\alpha)|^2 + \sum_{M\in (T\setminus\mathbb{F}_q^\times)/_w} (q^2+q)\,|\psi(M)+\psi(M^w)|^2$$

$$= (q+1)^2(q-1) + (q^2-1)(q-1) + \frac{1}{2}(q^2+q)\Big(\underbrace{\sum_{M\in T\setminus\mathbb{F}_q^\times} (\psi(M)+\psi(M^w))(\psi(M^{-1})+\psi(M^{-w}))}\Big)$$

If $M\in\mathbb{F}_q^\times \Rightarrow \alpha^{1-w} = \alpha^{w-1} = 1$

$$= \sum_{M\in(T\setminus\mathbb{F}_q^\times)} \left(1+1+\psi(M^{1-w})+\psi(M^{w-1})\right)$$

$$= \sum_{M\in(T\setminus\mathbb{F}_q^\times)} 2 + \underbrace{\sum_{M\in T} \psi(M^{1-w})+\psi(M^{w-1})}_{0} - \sum_{M\in\mathbb{F}_q^\times} 2$$

The sum of the $q-1$ roots of unity equals 0.

$$= 2\left((q-1)(q-2) - q - 1\right) = 2(q-1)(q-3)$$

$= (q+1)^2(q-1) + (q^2-1)(q-1) + \frac{1}{2}(q^2+q)\,2(q-1)(q-3) = (q^2-1)(q^2-q) = |Gl_2(\mathbb{F}_q)| \Rightarrow \psi^G$ is simple.

1.5. Type IV characters of $Gl_2(\mathbb{F}_q)$

$\theta : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ homomorphism of groups.

Given that $[\mathbb{F}_{q^2} : \mathbb{F}_q] = 2 \Rightarrow \mathbb{F}_{q^2}$ is a 2-dimensional $\mathbb{F}_q$-vector space.

$\Rightarrow \mathbb{F}_{q^2}^{\times}$ can be identified with a commutative semi simple subgroup of $Gl_2(\mathbb{F}_q)$ which we will denote by C. In fact we can identify $\left\{ \begin{pmatrix} 0 & -\varepsilon\varepsilon^q \\ 1 & \varepsilon+\varepsilon^q \end{pmatrix} \right\} \cong C \setminus \mathbb{F}_{q}^{\times}$

$\Rightarrow \theta$ can be seen as a character on C. and therefore we can induce a character on G :

$$\theta^G = ind_C^G(\theta).$$

Now let $\mu: \mathbb{F}_q^{\times} \to \mathbb{C}^{\times}$ as in Type I characters and $\lambda: \mathbb{F}_q^{\times} \to \mathbb{C}^{\times}$ a non trivial homomorphism

$\Rightarrow$ We denote by (μ,λ) the character on ZU given by :

$$(\mu,\lambda)\begin{pmatrix} \alpha & \alpha\beta \\ 0 & \alpha \end{pmatrix} = \mu(\alpha)\lambda(\beta)$$

Center of $Gl_2(\mathbb{F}_q)$

and we consider the induced character $(\mu,\lambda)^G := ind_{ZU}^G(\mu,\lambda)$

A computation similar to the previous cases give us the following values for these two characters on the conjugacy classes of $Gl_2(\mathbb{F}_q)$:

χ	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$	$A=\begin{pmatrix} 0 & -\varepsilon\varepsilon^q \\ 1 & \varepsilon+\varepsilon^q \end{pmatrix}$
θ^G	$(q^2-q)\theta(\alpha)$	0	0	$\theta(A)+\theta(F(A))$
$(\mu,G)^G$	$(q^2-1)\mu(\alpha)$	$-\mu(\alpha)$	0	0

That is, we used the induced character formula $\theta^G(A) = \frac{1}{|C|}\sum_{\substack{M\in Gl_2(\mathbb{F}_q) \\ M^{-1}AM\in C}} \theta(M^{-1}AM)$, $(\mu,\lambda)^G(A) = \frac{1}{|ZU|}\sum_{\substack{M\in Gl_2(\mathbb{F}_q) \\ M^{-1}AM\in ZU}} (\mu,\lambda)(M^{-1}AM)$

Frobenius reciprocity $\Rightarrow \theta^G$ appears in the character $(res\,\theta,\lambda)^G$ where $res\,\theta$ is the restriction of θ to $\mathbb{F}_p^{\times}$

$\Rightarrow$ We define $\theta' = (res\,\theta,\lambda)^G - \theta^G$. These characters, for $\theta \neq \theta\circ F$, are those of Type IV

Using the table above we find that :

χ	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$	$A=\begin{pmatrix} 0 & -\varepsilon\varepsilon^q \\ 1 & \varepsilon+\varepsilon^q \end{pmatrix}$
θ' $\theta\neq\theta\circ F$	$(q-1)\theta(\alpha)$	$-\theta(\alpha)$	0	$-\theta(A)-\theta(A^q)$

In a similar way as in Type III characters we can show that these characters are irreducible.

¿How many characters are of this type?. One sees from the character table that the value of these characters only depend on the morphisms $\theta: \mathbb{F}_{q^2}^{\times} \to \mathbb{C}^{\times}$ such that $\theta \neq \theta\circ F$

$\Rightarrow \theta$ must send the generator of $\mathbb{Z}/q^2-1\mathbb{Z}$ to a primitive q^2-1 roots of unity

$\Rightarrow$ there are $q^2-1-(q-1)$ possibilities for the morphism θ. However, notice that $\theta'(A) = \theta'(F(A))$ for the representative of each conjugacy class $\Rightarrow \frac{q^2-q}{2}$ possible characters

Part II - The E-polynomial for the character variety $\mathcal{M}_2$

2.1 The character variety $\mathcal{M}_2$ and its E-polynomial

$$\mathcal{U}_2 = \{ \rho \in \mathrm{Hom}(\pi_1(\Sigma \setminus \{p\}), Gl_2(\mathbb{C})) \mid \rho(\gamma) = \xi_2^{-1} Id_2 \} = \{ (A_1, B_1, \dots, A_g, B_g) \in Gl_2(\mathbb{C})^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] = \xi_2^{-1} Id_2 \}$$

↓ Primitive root of unity so $\xi_2 = -1$

$$\Rightarrow \boxed{\mathcal{M}_2 = \mathcal{U}_2 // Gl_2(\mathbb{C})}$$

Hausel - Rodriguez Villegas, 08' showed that $E(\mathcal{M}_2, q) = (q-1)\sum_{\chi \in Irr(Gl_2(\mathbb{F}_q))} \frac{|Gl_2(\mathbb{F}_q)|^{2g-2}}{\chi(Id_2)^{2g-1}} \chi(-Id_2)$.

So we use the character table of $Gl_2(\mathbb{F}_q)$, that we already studied to compute this polynomial.

2.2 Computation E-polynomial of $\mathcal{M}_2$

We recall that $|Gl_2(\mathbb{F}_q)| = (q^2-1)(q^2-q)$

$$\Rightarrow E(\mathcal{M}_2, q) = (q-1)\sum_{\chi \in Irr(Gl_2(\mathbb{F}_q))} \frac{((q^2-1)(q^2-q))^{2g-2}}{\chi(Id_2)^{2g-1}} \chi(-Id_2) = (q-1)((q^2-1)(q^2-q))^{2g-2} \Big[\underbrace{\sum_{\chi \in Type I} \frac{\chi(-Id_2)}{\chi(Id_2)^{2g-1}}}_{[1]} + \dots + \underbrace{\sum_{\chi \in Type IV} \frac{\chi(-Id_2)}{\chi(Id_2)^{2g-1}}}_{[4]} \Big]$$

[1] $\displaystyle\sum_{\chi \in Type I} \frac{\chi(-Id_2)}{\chi(Id_2)^{2g-1}} = \sum_{j=1}^{q-1} \frac{\mu_j \circ \det(-Id_2)}{\mu_j \circ \det(Id_2)^{2g-1}} = \sum_{j=1}^{q-1} \frac{\mu_j(1)}{(\mu_j(1))^{2g-1}} = \sum_{j=1}^{q-1} 1 = q-1$

[2] $\displaystyle\sum_{\chi \in Type II} \frac{\chi(-Id_2)}{\chi(Id_2)^{2g-1}} = \sum_{j=1}^{q-1} \frac{\psi^G_{\mu_j} - \mu_j \circ \det(-Id_2)}{(\psi^G_{\mu_j} - \mu_j \circ \det(Id_2))^{2g-1}} = \sum_{j=1}^{} \frac{q}{q^{2g-1}} = \frac{q(q-1)}{q^{2g-1}}$

[3] $\displaystyle\sum_{\chi \in Type III} \frac{\chi(-Id_2)}{\chi(Id_2)^{2g-1}} = \frac{1}{2}\sum_{j=1}^{(q-1)(q-2)} \frac{(q+1)\psi_j(-Id_2)}{((q+1)\psi_j(Id_2))^{2g-1}} = \frac{1}{2}\frac{1}{(q+1)^{2g-2}} \sum_{j=1}^{(q-1)(q-2)} \psi_j(-Id_2)$

$$= \frac{1}{2}\frac{1}{(q+1)^{2g-2}} \left| 2\left(\frac{q-1}{2}\right)\left(\frac{q-1}{2}-1\right) - 2\left(\frac{q-1}{2}\right)\left(\frac{q-1}{2}\right) \right)$$

($\psi_1 \to 1, \psi_2 \to 1$; $\psi_1 \to -1, \psi_2 \to -1$ | $\psi_1 \to 1, \psi_2 \to -1$; $\psi_1 \to -1, \psi_2 \to 1$)

Remember that ←

$\varphi(-1) = \varphi(e^{\frac{2\pi i}{q-1}})^{\frac{q-1}{2}} = (e^{\frac{2\pi i k}{q-1}})^{\frac{q-1}{2}} = e^{\pi i k}$

So it all depends on the parity of k

$$= -\frac{1}{2}\frac{1}{(q+1)^{2g-2}} \cdot (q-1)$$

[4] $\displaystyle\sum_{\chi \in Type IV} \frac{\chi(-Id_2)}{\chi(Id_2)^{2g-1}} = \frac{1}{2}\sum_{j=1}^{q^2-q} \frac{(q-1)\theta_j(-1)}{((q-1)\theta_j(Id_2))^{2g-1}} = \frac{1}{2}\frac{1}{(q-1)^{2g-2}} \sum_{j=1}^{q^2-q} \theta_j(-1) = \frac{1}{2}\frac{1}{(q-1)^{2g-2}} -(q-1)$

There are $\frac{q^2-1}{2}$ sending

There are $\frac{q^2-1}{2}$ morphism sending -1 to 1 and $\frac{q^2-1}{2}$ sending 1 to 1. However, all the morphism sending $e^{\frac{2\pi i}{q^2-1}}$ to a $q-1$ root of unity are such that sent -1 to 1

$$\Rightarrow E(\mathcal{M}_2, q) = \left((q^2-1)(q^2-q)\right)^{2g-2} \left(q-1 + \frac{q-1}{q^{2g-2}} - \frac{(q-1)}{(q+1)^{2g-2}} - \frac{1}{2} \frac{1}{(q-1)^{2g-3}} \right)$$

$$= (q-1)^{2g} (q^2-1)^{2g-2} q^{2g-2} + (q-1)^{2g} (q^2-1)^{2g-2} - \frac{1}{2} q^{2g-2} (q-1)^{4g-2} - \frac{1}{2} q^{2g-2} (q+1)^{2g-2} (q-1)^{2g}.$$