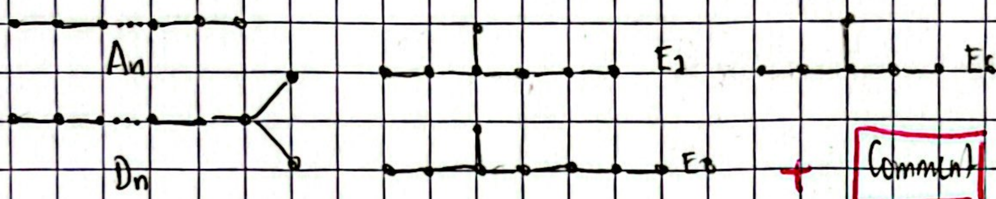
Gabriel's Theorem

Theorem (Gabriel, 1972) If Q is a connected quiver with underlying graph Γ, then there are only finitely many indecomposable representations (up to isomorphism) if and only if Γ is any of the Dynkin diagrams:



+ Comment

Why indecomposables are important?

Agenda

1. Some Generalities about Quiver representations

We fix $K = \mathbb{C}$.

- A **quiver** is a finite oriented graph, $Q = (V, E, h, t)$ $\quad h, t : E \to V$
- A **non trivial path** is a sequence of edges: $\bullet \xrightarrow{p_1} \bullet \xrightarrow{p_2} \cdots \bullet \xrightarrow{p_m} \bullet$ $\quad h(p_k) = t(p_{k+1})$
- The **path algebra** KQ is the K-algebra whose product is given by

$$x \cdot y = \begin{cases} \text{obvious composition}, & \text{if } h(y) = t(x) \\ 0, & \text{otherwise} \end{cases}$$

with orthogonal idempotents being e_v (the trivial paths) and $1 = \sum_{v \in V} e_v$.

- A **representation** of Q is given by $K^{d_{v_1}} \xrightarrow{\varphi_p} K^{d_{v_2}}$ $\to$ Dimension vector $(d_v)_v \in \mathbb{N}^{|V|}$
- A **morphism of representation** is a tuple $(\theta_v)_{v \in V}$ s.t.

$$\begin{tikzcd} K^{d_{v_1}} \arrow[r, "\varphi_p"] \arrow[d, "\theta_{v_1}"'] \arrow[dr, phantom, "\#"] & K^{d_{v_2}} \arrow[d, "\theta_{v_2}"] \\ K^{d'_{v_1}} \arrow[r, "\varphi'_p"'] & K^{d'_{v_2}} \end{tikzcd}$$

- A representation of Q is **indecomposable** iff $W \neq W_1 \oplus W_2 \oplus \cdots \oplus W_k$

$\quad \hookrightarrow$ Example. $\circlearrowleft \mathbb{C}^2$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Lemma: The category $\mathrm{Rep}(Q)$ is equivalent to KQ-Mod

Some useful definitions:

- (Euler Form) $\langle \alpha, \beta \rangle = \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{p \in E} \alpha_{t(p)} \beta_{h(p)}$ $\quad \forall \alpha, \beta \in \mathbb{Z}^{|V|}$ (Depends on the orientation)
- (Tits form) $q(\alpha) = \langle \alpha, \alpha \rangle$
- (Symmetric Bilinear form) $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$

Theorem (The standard resolution) Let X be an KQ-module. The exact sequence

$$0 \to \bigoplus_{\rho \in E} KQ\, c_{h(\rho)} \otimes_K c_{t(\rho)} X \xrightarrow{\partial_1} \bigoplus_{v \in V} KQ\, c_v \otimes_K c_v X \xrightarrow{\partial_0} X \to 0$$

is a projective resolution for $\partial_0(p \otimes x) = px$ and $\partial_1(p \otimes x) = p\rho \otimes x - p \otimes \rho x$.

__Corollaries:__

i) $\mathrm{Ext}^i(X, Y) = 0 \quad \forall Y,\ i \geq 2$.

ii) If X, Y are finite dimensional $\Rightarrow \dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y) = \langle \dim X, \dim Y \rangle$

Idea of the proof: Apply the functor $\mathrm{Hom}(-, Y)$ to the standard resolution.

iii) If X is finite dimensional $\Rightarrow \dim \mathrm{End}(X) - \dim \mathrm{Ext}^1(X,X) = q(\dim X)$

Projective Module

$$\begin{tikzcd} & N \arrow[d, two heads] \\ P \arrow[ur, dashed, "\exists h"] \arrow[r] & M \end{tikzcd}$$

Gabriel's theorem

Theorem (Gabriel, 1972): If Q is a connected quiver with underlying graph Γ, then there are only finitely many indecomposable representations if and only if Γ is one of the Dynkin diagrams:

2. Bricks $k = \mathbb{C}$, All modules are assumed to be finite dimensional. $\}(K)$

Let Q be a quiver; kQ its path algebra and let X be a kQ-module.

Recall that X is indecomposable $\iff$ $\mathrm{End}(X)$ is a local algebra
$\iff \mathrm{End}(X) = k\, \mathrm{Id}_X + \mathrm{rad}(\mathrm{End}(X))$.

Definition $\to$ We say that X is a brick if $\mathrm{End}(X) = k$. Thus, a brick is indecomposable.
$(0 \to X \to E \to Y \to 0)$

When does the converse hold?

Lemma: Suppose X, Y are indecomposable. If $\mathrm{Ext}^1(Y, X) = 0 \Rightarrow$ any $\theta : X \to Y$ non-zero is a monomorphism or an epimorphism.

Proof: We have exact sequences $0 \to \mathrm{Im}(\theta) \to Y \to \mathrm{coker}(\theta) \to 0 : \xi$

$0 \to \ker(\theta) \to X \to \mathrm{Im}(\theta) \to 0$

From the last exact sequence, we get an exact sequence (Applying $\mathrm{Hom}(\mathrm{coker}\,\theta, -)$)

$$\cdots \to \mathrm{Ext}^1(\mathrm{coker}\,\theta, \ker\theta) \to \mathrm{Ext}^1(\mathrm{coker}\,\theta, X) \xrightarrow{f} \mathrm{Ext}^1(\mathrm{coker}\,\theta, \mathrm{Im}\,\theta) \to 0$$

$\Rightarrow \xi = f(\eta)$ for some $\eta \in \mathrm{Ext}^1(\mathrm{coker}\,\theta, X)$. (Recall that $\mathrm{Ext}^1(A, B)$ classifies exact sequences $0 \to B \to E \to A \to 0$)

Thus, we have: a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\#"] & Z \arrow[r] \arrow[d, "\gamma"] \arrow[dr, phantom, "\#"] & \mathrm{coker}(\theta) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Im}(\theta) \arrow[r, "\delta"] & Y \arrow[r] & \mathrm{coker}(\theta) \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ the exact sequence $0 \to X \xrightarrow{\alpha \oplus \beta} Z \oplus \mathrm{Im}(\theta) \xrightarrow{\gamma - \delta} Y \to 0$ is exact

so it splits since $\mathrm{Ext}^1(Y, X) = 0$. If $\mathrm{Im}(\theta) \neq 0 \Rightarrow$ By the unique decomposition theorem a.k.a the Krull-Schmidt theorem, either X or Y is a summand of $\mathrm{Im}(\theta)$. But, If θ is neither monomorphism nor epimorphism $\Rightarrow$ $\dim \mathrm{Im}(\theta) < \dim X, \dim Y$ which is a contradiction.

Corollary: If X is indecomposable with no self-extensions $(\mathrm{Ext}^1(X,X) = 0)$, then X is a brick.

Lemma 2: If X is indecomposable, not a brick, then X has a submodule which is a brick with self-extensions.

Proof: It suffices to prove that if X is indecomposable and not a brick then there is a proper submodule $U \subset X$ which is indecomposable and with self-extensions. If U is not a brick we iterate.

Pick $\theta \in \mathrm{End}(X)$ with $I = \mathrm{im}\,\theta$ of minimal dimension $\neq 0$. By minimality, $\theta^2 = 0$. Let $\ker(\theta) = \bigoplus_{i=1}^{r} K_i$ with K_i indecomposable and let j such that the composition $\alpha : I \hookrightarrow \ker(\theta) \twoheadrightarrow K_j$ is non-zero. (The map $X \twoheadrightarrow I \xrightarrow{\alpha} K_j \hookrightarrow X$ has image $\mathrm{Im}(\alpha) \neq 0$ so α is a monomorphism by minimality.) (*)

Claim: $\mathrm{Ext}^1(I, K_j) \neq 0$ otherwise consider

$$\begin{array}{ccccccccc}
0 & \to & \bigoplus_{i=1}^{r} K_i & \to & X & \xrightarrow{\theta} & I & \to & 0 \\
 & & \downarrow & \# & \downarrow & \# & \| & & \\
0 & \to & K_j & \to & Y & \to & I & \to & 0
\end{array}$$

$Y = X / \bigoplus_{i \neq j} K_i$, $Y = K_j \oplus I$

$\Rightarrow$ the second sequence splits $\Rightarrow K_j$ is summand of X, a contradiction.

$\Rightarrow$ (*) Finally, the short exact sequence $0 \to I \to K_j \to \mathrm{coker}(\alpha) \to 0$ induces

$$\cdots \to \mathrm{Ext}^1(K_j, K_j) \to \mathrm{Ext}^1(I, K_j) \to \underbrace{\mathrm{Ext}^2(\mathrm{coker}(\alpha), K_j)}_{0} \to \cdots$$

$\Rightarrow \mathrm{Ext}^1(K_j, K_j) \neq 0$, take $U = K_j$.

2. Algebraic Geometry / the variety of representations

$\mathbb{A}^r := K^r +$ Zariski topology.

Definitions → • $U \subseteq \mathbb{A}^r$ is said to be locally closed if U is open in $\overline{U}$.
• A non-empty locally closed subset U is irreducible if any non-empty subset of U which is open in U is dense in U. For instance, $\mathbb{A}^r$ is irreducible.
• The dimension of a non-empty locally closed subset U is
$$\sup\{n \mid \exists\, Z_0 \subset Z_1 \subset \dots \subset Z_n \text{ irreducible subsets closed in } U\}.$$

Important remarks: • $\dim U = \dim \overline{U}$
• If $W = U \cup V \Rightarrow \dim W = \max\{\dim U, \dim V\}$

Let G be an algebraic group (variety having group structure where the multiplication and inversion maps are regular) acting on $\mathbb{A}^r$.

Some standard facts about group actions:
i) The orbits $\mathcal{O}$ are locally closed
ii) $\overline{\mathcal{O}} \setminus \mathcal{O}$ is the union of orbits of dimension strictly smaller than $\dim \mathcal{O}$.
iii) If $x \in \mathcal{O} \Rightarrow \dim \mathcal{O} = \dim G - \dim \mathrm{Stab}_G(x)$

Definitions → • Let Q be a quiver and $d \in \mathbb{N}^{|V|}$. We define
(V, E, h, t)

$$\mathrm{Rep}(Q,d) := \bigoplus_{e \in E} \mathrm{Mat}(d_{h(e)}, d_{t(e)}, K) \cong \mathbb{A}^r$$

for $r = \sum_{e \in E} d_{h(e)} d_{t(e)}$.

$GL(d) := \bigoplus_{v \in V} GL(d_v, K) \curvearrowright \mathrm{Rep}(Q,d)$ by conjugation. Explicitly:

For $M = (M_v)_{v \in V} \in GL(d)$ and $X := (\varphi_e)_{e \in E}$,
$$\Rightarrow M \cdot X = (M_{h(e)} \varphi_e M_{t(e)}^{-1})_{e \in E} \in \mathrm{Rep}(Q,d)$$

Proposition: i) $\mathcal{O}_X = GL(d) \cdot X = \{X' \in \mathrm{Rep}(Q,d) \mid X \cong X'\}$
ii) $\mathrm{Stab}_{GL(d)}(X) = \mathrm{Aut}(X)$

Lemma: $\dim Rep(Q,d) - \dim \mathcal{O}_\varphi = \dim End_{KQ}(\varphi) - q(d) = \dim Ext^1(\varphi,\varphi)$

Tits form: $q(d) = \langle d,d\rangle = \sum_{v\in V} d_v^2 - \sum_{\alpha\in E} d_{h(\alpha)} d_{t(\alpha)}$

Proof: We have that

$$\dim \mathcal{O}_\varphi = \dim GL(d) - \dim Stab_{GL(d)}(\varphi) = \dim GL(d) - \dim Aut(\varphi)$$

$\phi \neq GL(d) \subseteq \mathbb{A}^s$ is ~~non~~ empty and open in $\mathbb{A}^s$. $\Rightarrow$ so it is dense as $\mathbb{A}^s$ is irreducible
$s = \sum d_v^2$ $\Rightarrow \dim GL(d) = \dim(\mathbb{A}^s) = s = \sum_{v\in V} d_v^2$

Similarly, $Aut(\varphi)$ is non-empty and open in $End(\varphi)$ $\Rightarrow$ so it is dense and therefore $\dim Aut(\varphi) = \dim End(\varphi)$

$$\Rightarrow \dim Rep(Q,d) - \dim \mathcal{O}_\varphi = \sum_{\alpha\in E} d_{h(\alpha)} d_{t(\alpha)} - \sum_{v\in V} d_v^2 + \dim End(\varphi)$$

$$= \dim End(\varphi) - q(d) = Ext^1(\varphi,\varphi) \quad \left(\begin{array}{c}\text{Corollary (iii)}\\ \text{Section 1}\end{array}\right)$$

↳ this follows from the standard resolution (*)

Corollaries

~~Some corollaries of the lemma:~~

i) If $d \neq 0$ and $q(d) \leq 0$ $\Rightarrow$ there are infinitely many orbits in $Rep(Q,d)$.
Proof: $End(\varphi) \neq 0$ $\Rightarrow$ By the previous lemma $\dim Rep(Q,d) - \dim \mathcal{O}_\varphi > 0$. This implies the statement because if $Rep(Q,d) = \underbrace{\mathcal{O}_{\varphi_1} \cup \dots \cup \mathcal{O}_{\varphi_k}}_{\text{All the orbits of the action}} \Rightarrow \dim Rep(Q,d) = \max\{\mathcal{O}_{\varphi_i}\}$ which is a contradiction.

ii) $\mathcal{O}_\varphi$ is open $\Leftrightarrow Ext^1(\varphi,\varphi) = 0$.
Proof $Ext^1(\varphi,\varphi) = 0 \Leftrightarrow \dim \mathcal{O}_\varphi = \dim Rep(Q,d) \Leftrightarrow \dim \overline{\mathcal{O}_\varphi} = \dim Rep(Q,d) \Rightarrow \overline{\mathcal{O}_\varphi} = Rep(Q,d)$ since a proper closed of an irreducible subset has strictly smaller dimension. By definition of locally closed subset, $\mathcal{O}_\varphi$ is open in $Rep(Q,d)$.
Conversely, if $\mathcal{O}_\varphi$ is open in $Rep(Q,d)$ $\Rightarrow \overline{\mathcal{O}_\varphi} = Rep(Q,d)$ since $Rep(Q,d)$ is irreducible
$\Rightarrow \dim Rep(Q,d) = \dim \mathcal{O}_\varphi \Rightarrow Ext^1(\varphi,\varphi) = 0$.

iii) There is at most one module without self extensions of dimension d (up to isomorphism).
Proof If $\mathcal{O}_X \neq \mathcal{O}_Y$ are open $\Rightarrow \mathcal{O}_X \subseteq Rep(Q,d) \setminus \mathcal{O}_Y \Rightarrow \overline{\mathcal{O}_X} \subseteq Rep(Q,d)\setminus \mathcal{O}_Y$ which contradicts the irreducibility of $Rep(Q,d)$.

Lemma 2. If $\xi: 0 \to U \to X \to W \to 0$ is a non-split exact sequence, then $\mathcal{O}_{U\oplus W} \subseteq \overline{\mathcal{O}_X} \setminus \mathcal{O}_X$.

Proof: For each $v\in V$, identify U_v as a subspace of X_v. Choosing a basis for each U_v and then extending it to a basis of X_v, we can write

$$\varphi_\rho^X = \begin{pmatrix} \varphi_\rho^U & M_\rho \\ 0 & \varphi_\rho^W \end{pmatrix} \quad \text{where } M_\rho: W \to U.$$

Now consider the one-parameter subgroup $\mathbb{C}^* \to Gl(d,\mathbb{C})$

$\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \cdot \varphi_\alpha^X = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi_p^U & M_p \\ 0 & \varphi_p^W \end{pmatrix} \begin{pmatrix} 1/\lambda & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \varphi_p^U & \lambda M_p \\ 0 & \varphi_p^W \end{pmatrix}$$

$$\Rightarrow \lim_{\lambda \to 0} \begin{pmatrix} \varphi_p^U & \lambda M_p \\ 0 & \varphi_p^W \end{pmatrix} = \begin{pmatrix} \varphi_1^U & 0 \\ 0 & \varphi_p^W \end{pmatrix} \Rightarrow U \oplus W \in \overline{\mathcal{O}}_X$$

Finally, if we apply $Hom(-,U)$, we get:

$$0 \to Hom(W,U) \to Hom(X,U) \to Hom(U,U) \xrightarrow{f} Ext^1(W,U) \to coker(f) \to 0$$

$$\Rightarrow \dim Hom(W,U) - \dim Hom(X,U) + \dim Hom(U,U) - \dim(Im(f)) = 0$$
$$\Rightarrow \dim(Hom(U \oplus W, U)) - \dim Hom(X,U) = \dim Im(f)$$

But since ξ does not split, $f(id_U) = \xi \neq 0 \Rightarrow \dim Im(f) > 0$

$\Rightarrow Hom(U \oplus W, U) \not\cong Hom(X,U)$

$\Rightarrow U \oplus W \not\cong X$. $\blacksquare$

Corollaries

i) If $\mathcal{O}_X$ is an orbit in $Rep(Q,d)$ of maximal dimension and $X = U \oplus W$, then $Ext^1(V,U) = 0$

Proof. If there is a non-split extension $0 \to U \to E \to V \to 0$ then $\mathcal{O}_X \subsetneq \overline{\mathcal{O}_E} \setminus \mathcal{O}_E$ so $\dim \mathcal{O}_X < \dim \mathcal{O}_E$.

ii) If $\mathcal{O}_X$ is closed, then X is semisimple.

4. Dynkin and Euclidean diagrams

Some definitions...

- $\Gamma :=$ Finite graph with vertices $\{1, \dots, n\}$. We allow loops and multiple edges.

$\Rightarrow n_{ij} = n_{ji}$ Number of edges between vertices i and j.

- $q(\alpha) = \sum_{i=1}^{n} \alpha_i^2 - \sum_{i \leq j} n_{ij} \alpha_i \alpha_j$

- $(-,-)$ Symmetric bilinear form on $\mathbb{Z}^n$ given by $(\varepsilon_i, \varepsilon_j) = \begin{cases} -n_{ij} & (i \neq j) \\ 2 - 2n_{ii} & (i = j) \end{cases}$ for ε_i the i-th coordinate vector

- $q(\alpha) = \frac{1}{2}(\alpha, \alpha)$, $(\alpha, \beta) = q(\alpha + \beta) - q(\alpha) - q(\beta)$ $\boxed{\text{Comment}}$

⚠ If Γ is the underlying graph of a quiver $\Rightarrow q, (-,-)$ correspond to the Tits form and the symmetric bilinear form for quivers.

- We say that $\rightarrow q$ is positive (semi-) definite iff $q(\alpha) (\geq) > 0 \; \forall \alpha \in \mathbb{Z}^n$

- The radical of q is $\mathrm{rad}(q) = \{\alpha \in \mathbb{Z}^n \mid (\alpha, -) = 0\}$

- We have a partial ordering on $\mathbb{Z}^n$ given by $\alpha \leq \beta$ iff $\beta - \alpha \in \mathbb{N}^n$
- We say that $\alpha \in \mathbb{Z}^n$ is sincere if each component is non-zero.

Lemma. If Γ is connected and $\beta \geq 0$ is a non-zero radical vector $\Rightarrow \beta$ is sincere and q is positive-semi-definite. Moreover, for $\alpha \in \mathbb{Z}^n$, we have

$$q(\alpha) = 0 \Leftrightarrow \alpha \in \mathbb{Q}\beta \Leftrightarrow \alpha \in \mathrm{rad}(q)$$

Proof: By assumption, $0 = (\beta, \varepsilon_i) = (\varepsilon_i, \beta) = (2 - 2n_{ii})\beta_i - \sum_{j \neq i} n_{ij} \beta_j$.

(Sincere) If $\beta_i = 0 \Rightarrow \sum_{j \neq i} n_{ij} \beta_j = 0$ and since each term is greater than zero we find that $\beta_j = 0$ whenever it is an edge $i - j$. Since Γ is connected, it follows that $\beta = 0$, a contradiction $\Rightarrow \beta$ is sincere.

Every vertex is connected to another

(Positive semi definite) Now, we show that q is positive semi definite. Note that, for $\alpha \in \mathbb{Z}^n$

$$0 \leq \sum_{i<j} n_{ij} \frac{\beta_i \beta_j}{2} \left(\frac{\alpha_i}{\beta_i} - \frac{\alpha_j}{\beta_j} \right)^2$$

$$= \sum_{i<j} n_{ij} \frac{\beta_j}{2\beta_i} \alpha_i^2 - \sum_{i<j} n_{ij} (\alpha_i \alpha_j) + \sum_{i<j} n_{ij} \frac{\beta_i}{2\beta_j} \alpha_j^2 = \sum_{i \neq j} n_{ij} \frac{\beta_j}{2\beta_i} \alpha_i^2 + \sum_{i<j} n_{ij} \alpha_i \alpha_j$$

By the previous identity $= \sum_i (2-2n_{ii})\cancel{\beta_i}\frac{1}{2\cancel{\beta_i}}\alpha_i^2 - \sum_{i<j} n_{ij}\alpha_i\alpha_j = q(\alpha) = \sum_i (2-2n_{ii})\frac{\alpha_i^2}{2} + \sum_{i<j} n_{ij}\alpha_i\alpha_j = q(\alpha)$ (9)

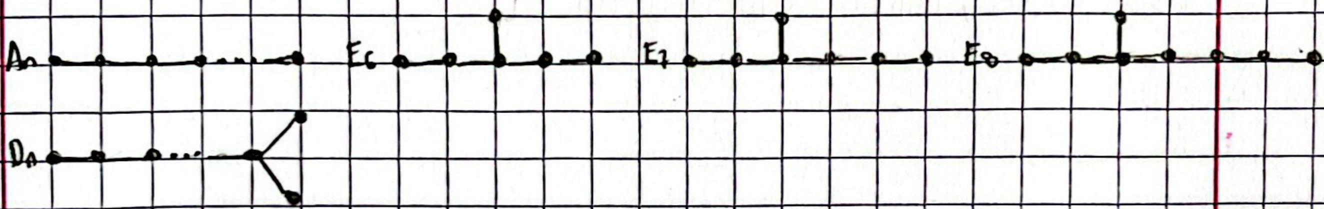
$\Rightarrow q$ is positive semi-definite.

Finally, if $q(\alpha)=0 \Rightarrow (\alpha_i/\beta_i - \alpha_j/\beta_j)^2 = 0$ and $\alpha_i/\beta_i = \alpha_j/\beta_j$ whenever there is an edge $i - j$. Since Γ is connected it follows that $\alpha \in \mathbb{Q}\beta$. It also follows that $\alpha \in rad(q)$ since $\beta \in rad(q)$ by assumption and if $\alpha \in rad(q)$ then certainly $q(\alpha)=0$.

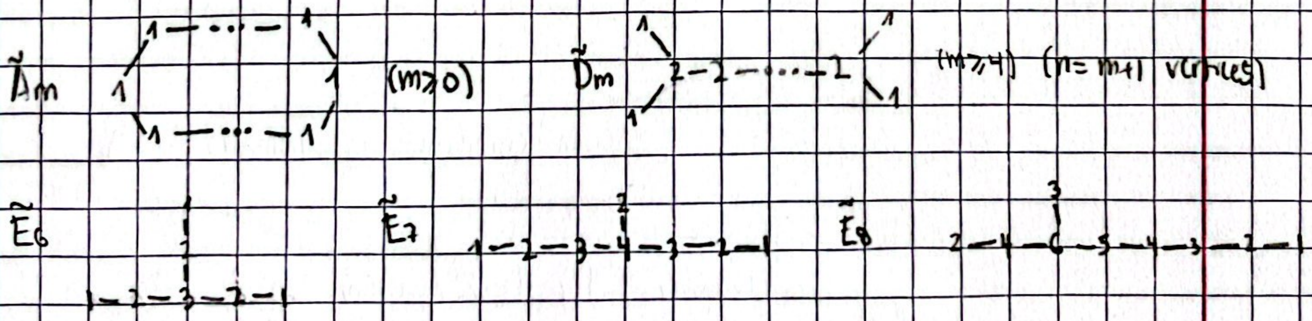
Classification of Graphs

Suppose Γ is connected.

(1) If Γ is Dynkin then q is positive definite. By definition, the Dynkin diagrams are:



(2) If Γ is Euclidean, then q is positive semi-definite and $rad(q) = \mathbb{Z}\delta$. By definition, the following are the Euclidean diagrams, where each vertex is marked with the value of δ_i. Note that δ is sincere and $\delta \geq 0$.



Note that $\tilde{A}_0$ has one vertex and one loop, and $\tilde{A}_1$ has two vertices joined by two edges.

(3) Otherwise, there is a vector $\alpha \geq 0$ with $q(\alpha) < 0$ and $(\alpha, \varepsilon_i) \leq 0$ $\forall i$.

Proof: (2) Note that $(\beta, \delta) = \sum_{i=1}^m \beta_i(\varepsilon_i, \delta) = \sum_{i=1}^m \beta_i\left(2\delta_i - \sum_{i \neq j} n_{ij}\delta_j\right)$ and one can check,

For each one of the euclidean graphs, that $(2\delta_i - \sum_{j\neq i} n_{ij}\delta_j) = 0$

$\Rightarrow \delta$ is radical and, by the lemma, q is positive semi-definite. Finally, note that for each one of the euclidean graphs, there exists i for which $\delta_i = 1 \Rightarrow rad(q) = \mathbb{Q}\delta \cap \mathbb{Z}^n = \mathbb{Z}\delta$.

(1) Embed the Dynkin diagram in the corresponding Euclidean diagram $\tilde{\Gamma}$ and note that q is strictly positive on non-zero non-sincere vectors. (Use formula for $q(\alpha)$ used to proof the lemma)

(3) One can show that Γ has a Euclidean subgraph Γ', with radical vector δ.

Vertices Γ = Vertices Γ' $\Rightarrow$ Take $\alpha = \delta$

let $i \notin \Gamma'$ connected to Γ' $\Rightarrow$ Take $\alpha = 2\delta + \varepsilon_i$

+ Use relation $(\beta, \varepsilon_i) = (2 - 2n_{ii})\beta_i - \sum_{j \neq i} n_{ij}\beta_j$

Comment/write: In summary, Γ is Dynkin $\Leftrightarrow$ q is positive definite

$\boxed{\text{Extending vertices}}$ If Γ is euclidean, a vertex v is called an extending vertex if $\delta_v = 1$.

Note that: i) There always is an extending vertex

ii) The graph obtained by deleting it is the corresponding Dynkin diagram.

From now on, we assume that Γ is Dynkin or Euclidean $\Rightarrow$ q is positive semi-definite.

$\boxed{\text{Root Systems}}$

$\Delta = \{\alpha \in \mathbb{Z}^n \mid \alpha \neq 0, q(\alpha) \leq 1\}$ the set of roots

A root is said to be real if $q(\alpha) = 1$ and imaginary if $q(\alpha) = 0$.

Properties.

i) Each ε_i is a root (This is straight forward from the definition of $q(\alpha)$)

ii) If $\alpha \in \Delta \cup \{0\}$, so are $-\alpha$ and $\alpha + \beta$ with $\beta \in \mathrm{rad}(q)$.

Proof: $q(\beta \pm \alpha) = q(\beta) + q(\alpha) \pm (\beta, \alpha) = q(\alpha)$.

iii) $\{\text{imaginary roots}\} = \begin{cases} \phi, & \text{if } \Gamma \text{ is Dynkin} \\ \{r\delta \mid 0 \neq r \in \mathbb{Z}\} & \text{if } \Gamma \text{ is euclidean} \end{cases}$ (Lemma + Classification of graphs)

iv) Every root is positive or negative.

Proof. Let $\alpha = \alpha^+ - \alpha^-$ where $\alpha^+, \alpha^- \geq 0$ (non-zero + disjoint support)

Note that $(\alpha^+, \alpha^-) \leq 0$ $\Rightarrow$

$$1 \geq q(\alpha) = q(\alpha^+) + q(\alpha^-) - (\alpha^+, \alpha^-) \geq q(\alpha^+) + q(\alpha^-) \geq 0$$

Either α^+, α^- are imaginary $\Rightarrow$ either of them is sincere

$\Rightarrow$ The other one is zero, a contradiction.

v) If Γ is euclidean then $(\Delta\cup\{0\})/\overset{\mathrm{rad}(q)}{\overset{\shortparallel}{\mathbb{Z}\delta}}$ is finite

<u>Proof</u>: let v be an extending vertex. If α is a root with $\alpha_v=0$. Then $\delta-\alpha$ and $\delta+\alpha$ are roots such that $(\delta-\alpha)_v, (\delta+\alpha)_v \geq 0 \Rightarrow (\delta-\alpha), (\delta+\alpha)$ are positive roots.

$\Rightarrow \{\alpha\in\Delta\cup\{0\} \mid \alpha_v=0\} \subseteq \{\alpha\in\mathbb{Z}^n \mid -\delta\leq\alpha\leq\delta\}$ which is finite.

Now, if $\beta\in\Delta\cup\{0\} \Rightarrow \beta-\beta_v\delta$ belongs to the finite set $\{\alpha\in\Delta\cup\{0\} \mid \alpha_v=0\}$

vi) If Γ is Dynkin then Δ is finite

<u>Proof</u>: Embed Γ in the corresponding Euclidean graph $\tilde{\Gamma}$ with extending vertex v. We can now view a root $\alpha\in$ for Γ as a root for $\tilde{\Gamma}$ with $\alpha_v=0 \Rightarrow$ the result follows v).

5. Gabriel's Theorem

Here, we combine everything we have done so far in order to prove Gabriel's Theorem!

Theorem 1 Suppose Q is a quiver with underlying graph Γ Dynkin. The assignment $X \to \dim X$ induces a bijection between the isomorphism classes of indecomposable modules and the positive roots of q.

Proof. • If X is indecomposable $\Rightarrow$ X is a Brick. Otherwise, by Lemma 2 - Bricks Section there is $U \subseteq X$ a brick with self-extensions, and then

$$0 < q(\dim U) = \underset{\overset{|||}{K}}{\dim \mathrm{End}(U)} - \dim \mathrm{Ext}^1(U,U) \leq 0$$

(*) Recall that q coincides with the Tits form.

• If X is indecomposable $\Rightarrow$ X has no self-extensions and $\dim X$ is a positive root:

$$0 < q(\dim X) = 1 - \dim \mathrm{Ext}^1(X,X).$$

• If X, X' are two indecomposables with the same dimension vector $\Rightarrow X \cong X'$. This follows from Lemma 1 - Algebraic Geometry Section + Corollary 3, $\boxed{\text{Comment}}$

• If α is a positive root, then there is an indecomposable X with $\dim X = \alpha$. To see this, pick an orbit $\mathcal{O}_X$ of maximal dimension in $\mathrm{Rep}(Q,\alpha)$. If X decomposes, $X = U \oplus V \Rightarrow \mathrm{Ext}^1(U,V) = \mathrm{Ext}^1(V,U) = 0$ by Lemma 2 - Algebraic Geometry Section - Corollary 1.

$$\Rightarrow 1 = q(\alpha) = q(\dim U) + q(\dim V) + \underbrace{\langle \dim U, \dim V\rangle + \langle \dim V, \dim U\rangle}_{= (\dim U, \dim V)}$$

(*) Corollary 2 Section 1 Quivers

$$= q(\dim U) + q(\dim V) + \dim \mathrm{Hom}(U,V) + \dim \mathrm{Hom}(V,U) \geq 2$$

which is a contradiction.

Theorem 2 If Q is a connected quiver with graph Γ, then there are only finitely many indecomposable representations $\Leftrightarrow \Gamma$ is Dynkin.

Proof: If Γ is Dynkin $\Rightarrow$ The indecomposables correspond to the positive roots and there are only finitely many of them.

Conversely, suppose there are only finitely many indecomposable representations. As every module (representation) is a direct sum of indecomposables, it follows that

$\Rightarrow$ there are only finitely many isomorphism classes of modules of dimension α $\forall \alpha \in \mathbb{N}^{V}$.

Thm $\Rightarrow$ there are only finitely many orbits in $Rep(Q,d)$.

By lemma 1.1 - Algebraic Geometry section - Corollary 1, we have $q(d)>0$ for $0 \neq d \in \mathbb{N}^{|V|}$.

$\Rightarrow$ The classification of graphs implies that the underlying graph of Q is dynkin.