$h = h_{jk}\, dz_j \otimes d\bar{z}_k$

1. Kähler Manifolds

Alternative definition: X complex mfld + $h = g - i\omega$.

Definition → A Kähler manifold is a symplectic manifold (M,ω) + compatible integrable almost complex structure. The symplectic form ω is called Kähler form.

On a local complex chart $(U, z_1, \dots, z_n)$ we can write:

As an element in $\Omega^2(X,\mathbb{R}) \otimes \mathbb{C} = \Omega^2(X,\mathbb{C})$ → $\omega = \sum a_{jk}\, dz_j \wedge dz_k + \sum b_{jk}\, dz_j \wedge d\bar{z}_k + \sum c_{jk}\, d\bar{z}_j \wedge d\bar{z}_k$ $\quad a_{jk}, b_{jk}, c_{jk} \in \mathcal{C}^\infty(U,\mathbb{C})$

But $J^* dz_j = i\, dz_j$ and $J^* d\bar{z}_j = -i\, d\bar{z}_j$, then

$$\omega = J^*\omega = \sum (i \cdot i) a_{jk}\, dz_j \wedge dz_k + (i \cdot (-i))\, b_{jk}\, dz_j \wedge d\bar{z}_k + (-i)(-i)\, c_{jk}\, d\bar{z}_j \wedge d\bar{z}_k$$

$$\Rightarrow J^*\omega = \omega \iff a_{jk} = c_{jk} \ \forall j,k \iff \omega \in \Omega^{1,1}(X,\mathbb{C}).$$

Moreover,

$$0 = d\omega = \underbrace{\partial\omega}_{(2,1)\text{-form}} + \overbrace{\bar{\partial}\omega}^{(1,2)\text{ form}} \iff \begin{cases} \partial\omega = 0 \\ \bar{\partial}\omega = 0 \end{cases} \Rightarrow [\omega] \in H^{1,1}(X,\mathbb{C}).$$

Finally, since ω is real-valued, $\omega = \bar{\omega}$ so that

$\omega = \frac{i}{2} \sum h_{jk}\, dz_j \wedge d\bar{z}_k$ and $\bar{\omega} = -\frac{i}{2} \sum \overline{h_{jk}}\, d\bar{z}_j \wedge dz_k = \frac{i}{2} \sum \overline{h_{jk}}\, dz_k \wedge d\bar{z}_j = \frac{i}{2} \sum \overline{h_{kj}}\, dz_j \wedge d\bar{z}_k$

$\Rightarrow h_{jk} = \overline{h_{kj}}$ and the $n \times n$ matrix (h_{jk}) is hermitian. + positive definite as ω is non-degenerate.

Examples: • $\mathbb{C}^n$ + $\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$

• (Fubini-Study metric) $X = \mathbb{P}^n$

$\rho_j \in \mathcal{C}^\infty(U_j) := \dfrac{\|z\|^2}{|z_j|^2}$ and define $\omega_j := -\dfrac{1}{2\pi i} \partial\bar{\partial} \log(\rho_j)$

On $U_j \cap U_k$ we have

$$\log(\rho_j) - \log(\rho_k) = \log|z_k|^2 - \log|z_j|^2 = \log(z_k \bar{z}_k) - \log(z_j \bar{z}_j)$$

Now, $\partial\bar{\partial}(\log(z_j\bar{z}_j)) = 0 \Rightarrow \omega_j = \omega_k$ on $U_j \cap U_k$ $\Rightarrow$ The ω_j define a global real (1,1) form on $\mathbb{P}^n$:

$$\omega = -\frac{1}{2\pi i} \partial\bar{\partial} \log \|z\|^2.$$

We still need to check the positivity of ω or equivalently, that $(h_{jk}) \gg 0$.

One can check that $h_{\ell k} = \partial_\ell \bar{\partial}_k \rho_j = \frac{1}{\pi} \left(\delta_{\ell k} \dfrac{\|z\|^2 - |z_k|^2}{\|z\|^4} \right)$

$\Rightarrow \omega$ is Kähler.

In fact...

Proposition: If $\rho \in \mathcal{C}^\infty(X,\mathbb{R})$ is a strictly plurisubharmonic function $\left[\left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p)\right) \text{ is positive-definite}\right] \Rightarrow \omega = \frac{i}{2}\partial\bar{\partial}\rho$ is Kähler. Such a function is called Kähler potential.

2. Hodge Theory

Theorem (Hodge decomposition theorem) (X,ω) compact Kähler manifold. Then

$$H^k_{dR}(M,\mathbb{C}) \cong \bigoplus_{l+m=k} H^{l,m}_{\bar{\partial}}(M)$$

with $\overline{H^{l,m}(X,\mathbb{C})} \cong H^{m,l}(X,\mathbb{C})$.

I plan to discuss now on the techniques involved to show this theorem.

$M :=$ Real, oriented, compact n-dimensional manifold.
$g :=$ Riemannian metric on g. $\longrightarrow$ Induces a dual metric on T^*M, $\langle -,- \rangle$.

We extend $\langle -,- \rangle$ to $\Lambda^r T^*M$ by

$$\langle \alpha_1 \wedge \dots \wedge \alpha_r, \beta_1 \wedge \dots \wedge \beta_r \rangle = \det(\langle \alpha_i, \beta_j \rangle).$$

The Hodge $*$-operator $\rightarrow$ Given a k-form β, $*\beta$ is the unique $n-k$ form for which

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, dvol$$

$\forall \alpha \in \Omega^k(M,\mathbb{R})$.

$\Rightarrow$ We can define a product on the space of k-forms by

$$(\alpha,\beta) := \int_M \alpha \wedge *\beta = \int_M \langle \alpha, \beta \rangle \, dvol$$

Proposition: The operator $\delta : \Omega^{k+1}(M) \longrightarrow \Omega^k(M)$ defined by

$$\delta := (-1)^{nk+1} * d * \quad \rightsquigarrow \text{ If } n \text{ is even, } \delta = -*d*$$

is the formal adjoint of d with respect to the inner product $(-,-)$.

Proof: $(d\alpha, \beta) = \int_M d\alpha \wedge *\beta = \int_M \cancelto{0}{d(\alpha \wedge *\beta)} - (-1)^k \int_M \alpha \wedge d*\beta$

$$= -(-1)^k (-1)^{k(n-k)} \int_M \alpha \wedge **d*\beta = \int \alpha \wedge *\delta\beta$$
$$= (\alpha, \delta\beta).$$

We now define the Laplace-Beltrami operator of M.

$$\Delta : \Omega^k(M) \longrightarrow \Omega^k(M)$$
$$\alpha \longmapsto d\delta\alpha + \delta d\alpha$$

Some properties:
- Δ is self-adjoint, $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$.
- $[\Delta, d] = [\Delta, \delta] = [\Delta, *] = 0$
- $\Delta(\alpha) = 0 \iff d\alpha = \delta\alpha = 0$.

Definition $\rightarrow$ $\alpha \in \Omega^k(M)$ is said to be Harmonic if $\Delta\alpha = 0$.

Harmonic forms are very special because of the following:

Proposition: i) α is harmonic $\iff$ $\|\alpha\|^2$ is a local minimum in its de Rham class.

ii) In every de Rham class, there is at most one harmonic form.

Proof. i) Let $\alpha \in \Omega^k(M)$ such that $\|\alpha\|^2$ is a local minimum

$\Rightarrow \forall \beta \in \Omega^{k-1}(M)$, $f(t) = \|\alpha + t\, d\beta\|$ has a local minimum at $t=0$.

$\Rightarrow f'(t) = 2(\alpha, d\beta) = 2(\delta\alpha, \beta) = 0 \quad \forall \beta \in \Omega^{k-1}(M)$

$\Rightarrow \delta\alpha = 0$ and α is harmonic.

If α is harmonic $\Rightarrow \|\alpha + d\beta\|^2 \underset{\text{Using bilinearity}}{=} \|\alpha\|^2 + \|d\beta\|^2 + 2(\alpha, d\beta) \underset{\alpha \text{ harmonic}}{=} \|\alpha\|^2 + \|d\beta\|^2 \geq \|\alpha\|^2$.

Equality holds if $d\beta = 0$.

Theorem (Hodge) Let $\mathcal{H}^k(M)$ denote the vector space of harmonic k-forms. Then:

i) $\mathcal{H}^k(M)$ is finite dimensional and isomorphic to $H^k_{dR}(M, \mathbb{R})$

ii) $\Omega^k(M) = \Delta(\Omega^k(M)) \oplus \mathcal{H}^k(M) \simeq \mathcal{H}^k(M) \oplus d\,\Omega^{k-1}(M) \oplus \delta\,\Omega^{k+1}$.

Proof: Warner - Chapter 6.

We now consider the complex case... $\boxed{\text{3. Complex Hodge Theory}}$

We can extend the Hodge $*$-operator to $\Lambda^k T^* X \otimes \mathbb{C}$ so as the inner product $\langle -, - \rangle$ to an hermitian one $\langle -, - \rangle^h$.

$\Rightarrow$ We get a positive definite Hermitian inner product on $A^k(M, \mathbb{C})$.

$$(\alpha, \beta)^h = \int_M \langle \alpha, \beta \rangle^h \, d\mathrm{vol} = \int_M \alpha \wedge * \bar{\beta}$$

$\rightarrow$ For a Kähler manifold this is $\frac{\omega^n}{n!}$

Remark: $*$ maps (p,q) forms to $(n-q, n-p)$ type forms

Proposition: The operator $\partial^* := -*\bar{\partial}*$ ($\bar{\partial}^* := -*\partial*$) is the formal adjoint of ∂ (resp $\bar{\partial}$) relative to the hermitian product $(-,-)^h$. These are operators of bidegree $(-1,0)$ and $(0,-1)$ respectively.

~~We have as well analogs of the Laplace-Beltrami operators:~~

We have as well complex analogs of the Laplace-Beltrami operators:

$$\Delta_\partial := \partial\partial^* + \partial^*\partial$$

$$\Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

These send (p,q) forms to forms of the same bidegree. ($\Delta_\partial, \Delta_{\bar{\partial}}$ are of bidegree $(0,0)$).

$\Rightarrow \Delta_{\bar{\partial}}(\alpha) = 0$ iff $\Delta_{\bar{\partial}}(\alpha^{p,q}) = 0 \ \forall p,q$.

How do these Laplacians relate to the former one?

Theorem: X compact Kähler manifold $\Rightarrow \Delta = 2\Delta_{\bar{\partial}} = 2\Delta_\partial$

(Hint) **Proofs**: Involves the Kähler identities:

$$[\bar{\partial}^*, L] = i\partial$$
$$[\partial^*, L] = -i\bar{\partial}$$
$$[\Lambda, \bar{\partial}] = -i\partial^*$$
$$[\Lambda, \partial] = i\,\bar{\partial}^*$$

where $L: \Omega^k(X,\mathbb{C}) \rightarrow \Omega^{k+2}(X,\mathbb{C})$, $\alpha \mapsto \alpha \wedge \omega$ (Lefschetz operator) on Λ is its formal adjoint

$(-1)^k * L * =: \Lambda : \Omega^{k+2}(X,\mathbb{C}) \rightarrow \Omega^{k-2}(X,\mathbb{C})$, i.e. $(L\alpha,\beta)^h = (\alpha,\Lambda\beta)^h$.

$\Rightarrow$ If α is $\Delta_{\bar{\partial}}$-harmonic $\Rightarrow$ the components are $\Delta_{\bar{\partial}}$-harmonic $\Rightarrow \alpha$ is Δ-harmonic as well.

$$\Rightarrow \qquad \mathcal{H}^k(X) \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X)$$

Similarly, as in the real case, we can conclude $H^k_{dR}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X,\mathbb{C})$.

Important remark: this decomposition does not depend on the choice of Kähler structure.

Idea of the proof: $K^{p,q} \subseteq H^k_{dR}(X,\mathbb{C})$:= subspace of de Rham cohomology classes which are representable by a closed form of type (p,q).

We clearly have $H^{p,q}_{\bar{\partial}}(X,\mathbb{C}) \subseteq K^{p,q}$, now we show the converse.

$\omega :=$ closed form of type (p,q)

$\leadsto$ In a unique way, we can write $\omega = \alpha + \Delta\beta$ with α harmonic.

$\Rightarrow$ As $d\omega = 0$, $\overset{0}{0} = \underbrace{d\alpha}_{\alpha \text{ harmonic}} + d\underbrace{\Delta\beta^{p,q}}_{d\delta\beta^{p,q}+\delta d\beta^{p,q}}$, $\Delta\beta^{p,q}$ should be closed

$\Rightarrow [\delta d\beta^{p,q}] = 0$

$\Rightarrow \omega = \alpha^{p,q} + d\delta\beta^{p,q} \Rightarrow [\omega] = [\alpha^{p,q}]$ and $[\omega] \in H^{p,q}(X)$

$\Rightarrow K^{p,q} = H^{p,q}$. $K^{p,q}$ does not depend on the choice of the metric and this finishes the proof.

The latter also implies the following:

Corollary: $\overline{H^{p,q}(X,\mathbb{C})} = H^{q,p}(X,\mathbb{C})$ where complex conjugation acts naturally on $H^{p+q}(X,\mathbb{C}) = H^{p+q}(X,\mathbb{R}) \otimes \mathbb{C}$.

Proof: We clearly have $\overline{K^{p,q}} = K^{q,p}$.

4. Consequences of the Hodge decomposition theorem

$b^k(X) := \dim H^k_{dR}(X,\mathbb{C})$

$h^{p,q}(X) := \dim H^{p,q}_{dol}(X,\mathbb{C})$

$\xrightarrow[\text{decomposition theorem}]{\text{Hodge}}$
$$\begin{cases} b^k = \sum_{p+q=k} h^{p,q} \\ h^{p,q} = h^{q,p} \\ H^{k,k}(X,\mathbb{C}) \neq 0 \rightsquigarrow [\omega^k] \neq 0 \\ b^k \text{ is even if } k \text{ is odd} \end{cases}$$

Stokes theorem: $\omega^n = d(\eta \wedge \omega^{n-k})$, $\omega^k = d\eta$

$H^k(X,\mathbb{R})$

Write Hodge diamond on the board!

Topological invariants?

$H^{n,n}$

$H^{1,0} \quad H^{0,0} \quad H^{0,1}$

Example: $X = \mathbb{P}^n$.

Using Mayer-Vietoris one can compute that $\begin{cases} H^{2p}(X,\mathbb{C}) = \mathbb{C} \\ H^{2p+1}(X,\mathbb{C}) = 0 \end{cases}$

$\Rightarrow$ the Hodge diamond of the complex projective plane is

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ & & & \vdots & & & \\ 0 & & & & & & 0 \\ & & & 1 & & & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}$$

Which other example of Kähler manifold we know?

Theorem Every Riemann Surface is Kähler.

Proof: Let h be an arbitrary hermitian metric on X, Riemann surface.
(Such metrics always exist.
To see that X is Kähler, is enough to see that $\mathrm{Im}(h) = \frac{i}{2}(h-\bar{h})$ is
a closed 2-form or, more specifically a $(1,1)$ form.
The latter is straightforward as $h = \tilde{h}\, dz \otimes d\bar{z}$. To see that ω is
closed is a dimension argument as there cannot be differential forms
of degree greater than 2.

Is this the unique Kähler structure? $H^2(X,\mathbb{C}) = H^{1,1}(X,\mathbb{C}) = \mathbb{C}$
so any two Kähler forms are proportional.

The Hodge diamond of a Riemann surface is then

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

genus of X which gives topological information.

Further examples are:
- Complex tori: Take symplectic form induced from the euclidean space
- Complex submanifolds of Kähler manifolds

Another interesting application...

Lefschetz (1,1)-theorem The first chern class/degree gives a map from

$$\mathrm{Pic}(X) \longrightarrow H^2(X,\mathbb{Z}).$$

$\Rightarrow$ Every cohomology class on $H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Z})$ is that of a line bundle.

Proof: $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0 \rightsquigarrow \ldots \to \underset{\substack{\|\wr \\ H^1(X,\mathcal{O}_X^*)}}{\mathrm{Pic}(X)} \to H^2(X,\mathbb{Z}) \xrightarrow{i_*} H^2(X,\mathcal{O}_X)$

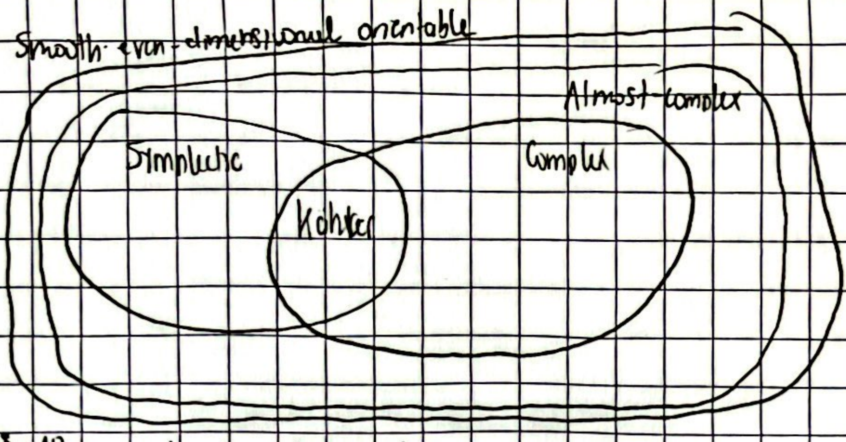
But $H^2(X,\mathcal{O}_X) = H^2(X,\Omega_X^0) \cong H^{0,2}(X,\mathbb{C})$.

One can show with some work that the diagram is commutative:

$$\begin{tikzcd}
& \check{H}^2(M,\mathbb{Z}) \arrow[r, "i_*"] \arrow[d] \arrow[dr, phantom, "\#"] & \check{H}^2(M,\mathcal{O}) \arrow[dd, "\cong"] \\
& \check{H}^2(M,\mathbb{C}) \arrow[d] & \\
H^{0,2} \oplus H^{1,1} \oplus H^{2,0} \arrow[r, phantom, "\cong"] & H^2_{dR}(X,\mathbb{C}) \arrow[r, "\pi^{(0,2)}"] & H^{0,2}_{d\bar{o}}(X,\mathbb{C})
\end{tikzcd}$$

$\Rightarrow$ Given $\gamma \in H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Z}) \Rightarrow i_*\gamma = 0 \Rightarrow \gamma$ is the chern class of some line bundle $L \in \mathrm{Pic}(X)$

Quick-summary of everything



- S^4, S^8, S^{10} are not almost complex
- Not every symplectic + complex manifold is Kähler. $\rightsquigarrow$ Kodaira-Thurston example
- There are symplectic manifolds that do not admit any complex structure

$$\begin{array}{c} S^1 \hookrightarrow M \\ S^1 \hookrightarrow P \\ \quad\;\; \downarrow \\ \quad\;\; T^2 \end{array}$$

- Given a complex structure is there always a symplectic structure? No
  Hopf surface $S^1 \times S^3$ not symplectic as $H^2(S^1 \times S^3) = 0$.
- Almost complex manifold either complex or symplectic? $\mathbb{CP}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$