The Moduli space of Elliptic Curves

Today's agenda:

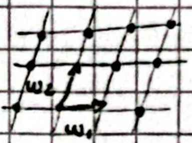
1. Elliptic curves : Definitions, Morphisms and the action $SL_2(\mathbb{Z}) \curvearrowright \mathcal{H}$
2. Orbifolds : Definitions, examples
3. The orbifold structure of the moduli space of elliptic curves.

$\boxed{\text{1. Elliptic Curves}}$

1.1 Basic definitions

$\omega_1, \omega_2 \in \mathbb{C}$ such that $Sp_{\mathbb{R}}\{\omega_1, \omega_2\} = \mathbb{C}$.

$\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ abelian group



$\Lambda \curvearrowright \mathbb{C}$ by translations. $\underset{\substack{\text{Free}\\+\\\text{Proper}}}{\Longrightarrow}$ $\mathbb{C}/\Lambda$ is a one-dimensional complex manifold (Riemann Surface).
Every R.S obtained this way is called Elliptic Curve.

- $\Lambda \cong \mathbb{Z}^2$ so that $\mathbb{C}/\Lambda \cong \mathbb{C}/\mathbb{Z}^2 \cong$. However different lattices may define different complex structures despite the underlying topological space being the same.
- $\mathbb{C}/\Lambda$ has a group structure compatible with the complex one.

¿When are two elliptic curves isomorphic?

Proposition: $\mathbb{C}/\Lambda \xrightarrow{f} \mathbb{C}/\Lambda'$ morphism $\left(\begin{smallmatrix}\text{holomorphic}\\+\\\text{group homo.}\end{smallmatrix}\right) \Rightarrow \exists\, a \in \mathbb{C}$ s.t. $a\Lambda \subset \Lambda'$ and $f(z \bmod \Lambda) = az \bmod \Lambda'$.

Proof:
$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{f}} & \mathbb{C} \\ \downarrow{\pi_\Lambda} & \# & \downarrow{\pi_{\Lambda'}} \\ \mathbb{C}/\Lambda & \xrightarrow[f]{} & \mathbb{C}/\Lambda' \end{array}$$

where $\tilde{f}$ is holomorphic and $\tilde{f}(0) = 0$.

Note that $\tilde{f}(z+\omega) - \tilde{f}(z) \in \Lambda'$ $\forall z \in \mathbb{C}, \omega \in \Lambda$ $\underset{\text{thm}}{\overset{\text{open Mapping}}{\Longrightarrow}}$ $\tilde{f}'(z+\omega) = \tilde{f}'(z)$

$\underset{\text{Liouville}}{\Longrightarrow}$ $\tilde{f}'(z)$ is constant $\Rightarrow \tilde{f}(z) = az + b \Rightarrow \tilde{f}(z) = az$.

Corollary: f is an isomorphism iff $\alpha\Lambda = \Lambda'$, for $\alpha \in \mathbb{C}^*$

1.2 The action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$.

$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \mid ad-bc=1 \right\}$ $\qquad \mathbb{H} = \{ \tau \in \mathbb{C} \mid Im(\tau) > 0 \}$

$SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}$ properly in the following way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau+b}{c\tau+d}$$

Observe that

$$\frac{1}{\omega_1}\Lambda = \mathbb{Z} + \mathbb{Z}\underbrace{\frac{\omega_2}{\omega_1}}_{\tau \in \mathbb{H}} \simeq \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

$\boxed{\text{Theorem}}$ $\forall \tau, \tau' \in \mathbb{H}$, $\mathbb{C}/\Lambda_\tau \cong \mathbb{C}/\Lambda_{\tau'}$ iff $\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that

$$\tau' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau+b}{c\tau+d}$$

Proof: $\begin{cases} f(\tau') = \alpha\tau' = a\tau + b \\ f(1) = \alpha = c\tau + d \end{cases}$ for some $a,b,c,d \in \mathbb{Z}$ $\Rightarrow$ $\tau' = \dfrac{a\tau+b}{c\tau+d}$.

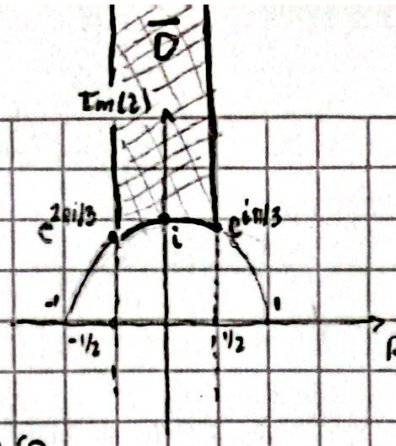
$\Rightarrow$ One can see then that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

$\Rightarrow$ $\mathbb{H}/SL_2(\mathbb{Z}) \xleftrightarrow[1:1]{}$ Isomorphism classes of elliptic curves

Let's get a better idea of how this quotient looks like...

1.3 The orbit space $\mathcal{H}/SL_2(\mathbb{Z})$

$\overline{D} = \{\tau \in \mathcal{H} \mid |Re(\tau)| \leq 1/2,\ |\tau| \geq 1\}$



Theorem:

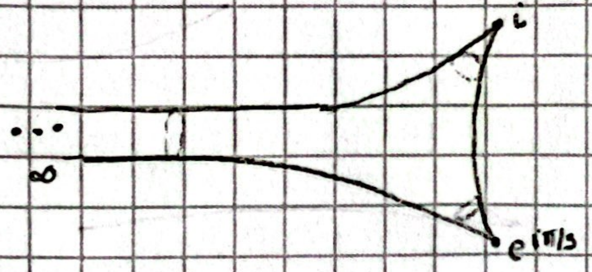
- $\forall\, O_\tau\ \exists\, w \in \overline{D}$ such that $w \in O_\tau$.
- $\forall\, z \neq w \in \overline{D}$ such that $\tau, w \in O_\tau \Rightarrow \begin{cases} Re(z) = 1/2 \Rightarrow w = z-1 \\ Re(z) = -1/2 \Rightarrow w = z+1 \\ |z| = 1 \Rightarrow w = -1/z \end{cases}$
- $\forall\, \tau \in \mathcal{H} \setminus \{i, e^{i\pi/3}, e^{2\pi i/3}\}$, $Stab(\tau) = \{\pm I\} \cong \mathbb{Z}/2\mathbb{Z}$.

  For the other 3 cases we have: $Stab(i) = \langle S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle \cong \mathbb{Z}/4\mathbb{Z}$

  $Stab(e^{2\pi i/3}) = \langle ST \rangle \cong \mathbb{Z}/6\mathbb{Z}$, $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

  $Stab(e^{i\pi/3}) = \langle TS \rangle \cong \mathbb{Z}/6\mathbb{Z}$

$\rightsquigarrow$



Through the theory of orbifolds we can study these spaces with singularities...

2. Orbifolds

2.1 Basic definitions

X a Hausdorff topological space and $\mathcal{U} = \{U_i\}_{i \in I}$ open cover closed by finite intersections

- Orbifold Chart $\longrightarrow \begin{cases} \bullet\ \tilde{U}_i \subseteq \mathbb{R}^n \text{ connected} \\ \bullet\ \Gamma_i \text{ finite group acting on } \tilde{U}_i \\ \bullet\ \varphi_i : \tilde{U}_i \to U_i \text{ continuous + surjective that induces } \tilde{U}_i/\Gamma_i \cong U_i \end{cases}$

- $\underline{\text{Embedding}} \longrightarrow$ Let $U_i \subseteq U_j$, $\begin{cases} \lambda_{ij} : \Gamma_i \hookrightarrow \Gamma_j \text{ monomorphism} + \operatorname{Ker}(\Gamma_i \to \operatorname{Diff}(\tilde{U}_i)) \xrightarrow{\cong} \operatorname{Ker}(\Gamma_j \to \operatorname{Diff}(\tilde{U}_j)) \\ \tilde{\varphi}_{ij} : \tilde{U}_i \hookrightarrow \tilde{U}_j \text{ smooth embedding} + \lambda_{ij}\text{-equivariant} \end{cases}$

- $\underline{\text{Orbifold Atlas}} \rightarrow$ An atlas $\mathcal{A}$ associated to $\mathcal{U}$ is a collection $\{(\tilde{U}_i, \Gamma_i, \varphi_i)\}$ locally compatible:

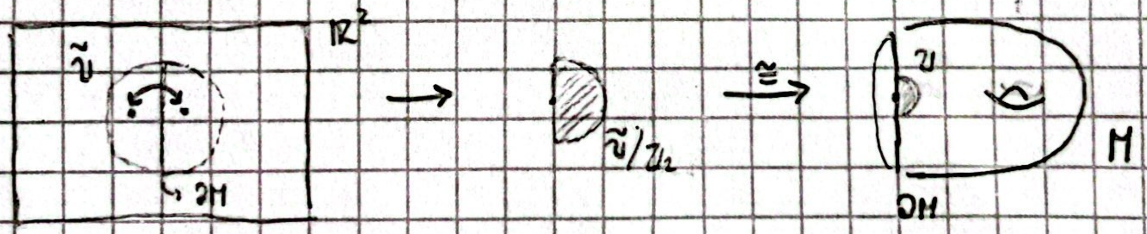
$$\exists\, U_k \subset U_{i_1} \cap U_{i_2} \text{ and a chart } (\tilde{U}_k, \Gamma_k, \varphi_k) \hookrightarrow (\tilde{U}_{i_1 i_2}, \Gamma_{i_1 i_2}, \varphi_{i_1 i_2})$$

- $\underline{\text{Equivalence of atlases}} \longrightarrow$ $\mathcal{A}_1 \sim \mathcal{A}_2 \iff$ They have common refinement. That is, there exists $\mathcal{A}'$ such that every orbifold chart of this atlas embedds in some orbifold chart of $\mathcal{A}_1$ and $\mathcal{A}_2$.

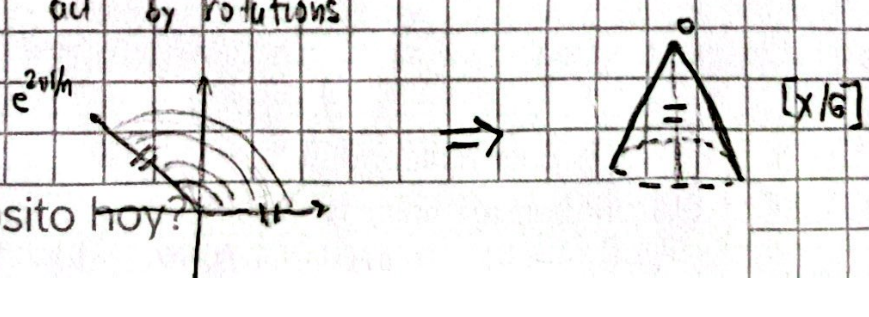
$\rightsquigarrow$ A smooth $\underline{n\text{-dimensional orbifold}}$ $\underline{X}$ is a Hausdorff top space X together with an equivalence class of orbifold atlases on it.

2.2 Examples

- An smooth manifold is an orbifold. If $\{(U_i, \phi_i)\}$ is an atlas $\Rightarrow$ $\tilde{U}_i = U_i$, $\Gamma_i = \{e\}$, $\varphi_i = Id$.

- 

- $X = \mathbb{C}$, $G = \mathbb{Z}_n$ act by rotations



¿Cuál es su propósito hoy?

3. The orbifold structure of the moduli space of elliptic curves

Theorem The orbit space $\mathcal{H}/SL_2(\mathbb{Z})$ has structure of 2-dimensional smooth orbifold. We denote this by $[\mathcal{H}/SL_2(\mathbb{Z})]$

Proof: $SL_2(\mathbb{Z}) \curvearrowright \mathcal{H}$ is proper $\Rightarrow \pi : \mathcal{H} \to \mathcal{H}/SL_2(\mathbb{Z})$

$\underset{1}{\Rightarrow}$ $\mathcal{H}/SL_2(\mathbb{Z})$ is a Hausdorff topological space.

$\underset{2}{\Rightarrow}$ let $\tilde{x} \in \pi^{-1}(x)$ $\exists$ $\tilde{V}_{\tilde{x}} \subseteq \mathcal{H}$ open neighborhood of $\tilde{x}$ such that $g \cdot \tilde{V}_{\tilde{x}} \cap \tilde{V}_{\tilde{x}} \neq \emptyset \Rightarrow g \in Stab(\tilde{x})$.

Since the stabilizers are finite, $\tilde{U}_{\tilde{x}} = \bigcap_{g \in Stab(\tilde{x})} g \cdot \tilde{V}_{\tilde{x}}$ is a $Stab(\tilde{x})$-invariant open neighborhood of $\tilde{x}$.

By shrinking, if necessary, we can furthermore assume that $\tilde{U}_{\tilde{x}}$ is connected.

$U_x := \pi(\tilde{U}_{\tilde{x}}) \subseteq \mathcal{H}/SL_2(\mathbb{Z})$ $\Rightarrow$

$$\begin{tikzcd}
\pi|_{\tilde{U}_{\tilde{x}}} : \tilde{U}_{\tilde{x}} \arrow[rr] \arrow[dr] & & \mathcal{H}/SL_2(\mathbb{Z}) \\
& \tilde{U}_{\tilde{x}}/Stab(\tilde{x}) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

$\Rightarrow \{U_x \mid x \in \mathcal{H}/SL_2(\mathbb{Z})\}$ is an open cover of $\mathcal{H}/SL_2(\mathbb{Z})$.

Each U_x has associated the orbifold chart $(\tilde{U}_{\tilde{x}}, \underset{\Gamma_{\tilde{x}}}{\overset{\shortparallel}{Stab(\tilde{x})}}, \pi|_{\tilde{U}_{\tilde{x}}})$

What remains? Closing by finite intersections the open cover and checking local compatibility between charts.

If $U_{x_1} \cap \dots \cap U_{x_n} \neq \emptyset \Rightarrow \exists g_1, \dots, g_n \in SL_2(\mathbb{Z})$ such that $\underbrace{g_1 \tilde{U}_{\tilde{x}_1} \cap \dots \cap g_n \tilde{U}_{\tilde{x}_n}}_{\tilde{U}_{\tilde{x}_1 \dots \tilde{x}_n}} \neq \emptyset$

Note that $\tilde{U}_{\tilde{x}_1 \dots \tilde{x}_n}$ is $\underbrace{g_1 Stab(\tilde{x}_1) g_1^{-1} \cap \dots \cap g_n Stab(\tilde{x}_n) g_n^{-1}}_{\Gamma_{\tilde{x}_1 \dots \tilde{x}_n}}$-invariant

$\Rightarrow (\tilde{U}_{\tilde{x}_1 \dots \tilde{x}_n}, \Gamma_{\tilde{x}_1 \dots \tilde{x}_n}, \pi|_{\tilde{U}_{\tilde{x}_1 \dots \tilde{x}_n}})$ gives an orbifold chart for $U_{x_1} \cap \dots \cap U_{x_n}$.

Finally, the compatibility of the charts is clear by construction. $\boxtimes$

Final remarks ...

The j-invariant of an elliptic curve is a complex number that characterizes the isomorphism class of an elliptic curve. This induces a bijection

$$\mathcal{H}/SL_2(\mathbb{Z}) \xrightarrow[j\text{-invariant}]{} \mathbb{C} \quad \text{coarser than } [\mathcal{H}/SL_2(\mathbb{Z})].$$

So, what do we gain with the orbifold point of view on $\mathcal{H}/SL_2(\mathbb{Z})$?

In the orbifold atlas given is encoded the information about the automorphism group of every elliptic curve. In this sense, "the moduli space" obtained via the j-invariant is coarser than the one we have build in this talk.