Moment map in Gauge Theory / The Moduli space of Flat connections

1. Connections on a principal bundle

G = compact Lie group, M = Differentiable manifold.

Definition $\to$ A principal G-bundle over M is a manifold P together with a smooth map $\pi: P \to M$ such that:

i) G acts freely (only the identity element fixes every element) on M (on the right)

ii) $M = P/G$ and π is the point-orbit projection.

iii) There is an open covering of M s.t. to each $\mathcal{U}$ in the covering $\exists\ \varphi_{\mathcal{U}}: \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times G$ diffeomorphism such that $\varphi_{\mathcal{U}}(p) = (\pi(p), s_{\mathcal{U}}(p))$ where $s_{\mathcal{U}}: \pi^{-1}(\mathcal{U}) \to G$ is a G-equivariant map $(s_{\mathcal{U}}(p \cdot g) = s_{\mathcal{U}}(p)\, g)$

We call M the base, P the total space and G the structure group. We usually represent a principal bundle as

$$\begin{array}{ccc} G & \hookrightarrow & P \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

Example (The Hopf Fibration):

$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$

$\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

$\mathbb{S}^1 \curvearrowright \mathbb{S}^3$ by complex multiplication $(\overset{e^{i\theta}}{\overset{\shortparallel}{z}}, (z_1, z_2)) = (e^{i\theta} z_1, e^{i\theta} z_2)$.

The quotient $\mathbb{S}^3/\mathbb{S}^1 \simeq \mathbb{CP}^1 \simeq \mathbb{S}^2$, this gives a $\mathbb{S}^1$-principal bundle

$$\begin{array}{ccc} \mathbb{S}^1 & \hookrightarrow & \mathbb{S}^3 \\ & & \downarrow \\ & & \mathbb{S}^2 \end{array}$$

For a fixed $p \in P$, the smooth map given by $\begin{array}{l} G \xrightarrow{\Psi_p} P \\ g \mapsto g \cdot p \end{array}$ is an embedding

$\Rightarrow$ Induces an injection $d_e\Psi_p: \mathfrak{g} \to T_pP$ and therefore an exact sequence:

This is just the infinitesimal action $d_e\Psi_p(\eta) = \frac{d}{dt}\Big|_{t=0} p\exp(t\eta)$

$$0 \to \mathfrak{g} \xrightarrow{d_e\Psi_p} T_pP \xrightarrow{d_p\pi} T_{\pi(p)}M \to 0$$

We refer to the vectors in $\underbrace{d_e\Psi_p(\mathfrak{g})}_{V_p} \subseteq T_pP$ as vertical tangent vectors of P.

We would like to choose a complement H_p s.t. $V_p \oplus H_p = T_pP$ of horizontal tangent vectors. This is equivalent to the choice of a linear map $A_p: T_pP \to \mathfrak{g}$ such that $A_p \circ d_e\Psi_p = \mathrm{id}_{\mathfrak{g}}$ (Take $H_p = \mathrm{Ker}\, A_p$).

Moreover, we require A_p to vary differentially, i.e. that the family $\{A_p\}$ defines

$A \in \Omega^1(P) \otimes \mathfrak{g}$

Example: The Maurer-Cartan form of a Lie group

$$G \hookrightarrow M\times G \xrightarrow{\pi_2} G$$
$$\pi_1 \downarrow$$
$$M$$

We define $\omega_{MC} \in \Omega^1(P)\otimes\mathfrak{g}$ fiberwise as:

$$(\omega_{MC})_{(p,g)} = d_{(p,g)}(L_{g^{-1}}\circ\pi_2) \quad \forall (p,g)\in M\times G.$$

- Let us check that ω_{MC} is smooth: Note that $\omega_{MC} = \pi_2^*(\omega_0)$ for $\omega_0 \in \Omega^1(G)\otimes\mathfrak{g}$ defined by $\omega_{0,g} = dL_{g^{-1}}$.

$\Rightarrow$ It is enough to show that ω_0 is smooth.
If $X\in\mathfrak{g}$ and X^L is the corresponding left-invariant vector field $\Rightarrow$ $\omega_{0g}(X^L) = X$ is constant and thus differentiable.
Every vector field Y on G can be written as a $\mathcal{C}^\infty(G)$-linear combination of left invariant vector fields $\Rightarrow \omega_0(Y)$ is smooth.

- Finally, let us show that $(\omega_{MC})_{(p,g)}\circ d\Psi_{(p,g)} = \mathrm{Id}_\mathfrak{g}$. In this particular case, $\Psi_{(p,g)}: G \to M\times G$, $h \mapsto (p, gh)$ $\Rightarrow (\omega_{MC})_{(p,g)}\circ d\Psi_{(p,g)} = d_e(L_{g^{-1}}\circ\pi_2\circ\Psi) = d_e\,\mathrm{Id} = \mathrm{Id}_\mathfrak{g}$.

Now, observe that
$$R_g^*\omega_{MC} = R_g^*\pi_2^*\omega_0 = \pi_2^* R_g^*\omega_0.$$

And, for all $h\in G$,
$$(R_g^*\omega_0)_h = \omega_{0,hg}\circ dR_g = d_{hg}L_{(hg)^{-1}}\circ d_hR_g = d_h L_{g^{-1}}\circ L_{h^{-1}}\circ R_g = d_h\,Ad\,g^{-1}\circ L_{h^{-1}} = Ad(g^{-1})\,\omega_{0,h^{-1}}$$

Linear map $\Omega^1(P,\mathfrak{g})\to\Omega^1(P,\mathfrak{g})$ induced by the adjoint representation $Ad(g^{-1}):\mathfrak{g}\to\mathfrak{g}$

Thus $R_g^*\omega_{MC} = \pi_2^*\,Ad(g^{-1})\,\omega_0 = Ad(g^{-1})\circ\omega_{MC}$.

This discussion prompts the following definition of connection

Definition $\to$ A connection on a principal G-bundle (P,π,M) is a 1-form $\omega\in\Omega^1(P)$ $A\in\Omega^1(P)\otimes\mathfrak{g}$ satisfying:
i) $A_p\circ d_e\Psi_p = \mathrm{id}_\mathfrak{g}$
ii) $R_g^*A = Ad(g^{-1})\circ A \quad \forall g\in G \iff dR_g(H_p) = H_{pg}$

2. The curvature of a connection

As with the "classic" differential forms we have an exterior differential
$$d: \Omega^k(P)\otimes\mathfrak{g} \longrightarrow \Omega^{k+1}(P)\otimes\mathfrak{g}$$
Relative to the choice of a basis $\{e_1,\dots,e_n\}$ for $\mathfrak{g}$, we can write
$$d\omega = d\omega_1\otimes e_1 + \dots + d\omega_n\otimes e_n$$
We can combine the Lie bracket together with the wedge product of forms to define

$$[\alpha,\beta] = \sum_{i,j} \alpha_i \wedge \beta_i \otimes [e_i,e_j] \in \Omega^{k+l}(P)\otimes\mathfrak{g} \quad \text{for } \begin{array}{l}\alpha\in\Omega^k(P)\otimes\mathfrak{g}\\ \beta\in\Omega^l(P)\otimes\mathfrak{g}.\end{array}$$ ③

composition of $\Omega^k(P)\otimes\mathfrak{g} \times \Omega^l(P)\otimes\mathfrak{g} \xrightarrow{\wedge} \Omega^{k+l}(P)\otimes\mathfrak{g}^2 \xrightarrow{[\,,\,]} \Omega^{k+l}(P)\otimes\mathfrak{g}$

<u>Definition</u> → The curvature form $F_A \in \Omega^2(E,\mathfrak{g})$ for the connection $\omega\in\Omega^1(P)\otimes\mathfrak{g}$ is defined by the equation

$$F_A = dA + \frac{1}{2}[A,A] \quad \text{or equivalently } d_A(A)$$

<u>Example</u>: Let us go back to the example of the Maurer-Cartan form! Let us show that

$$d\omega_0 + \frac{1}{2}[\omega_0,\omega_0] = 0$$

It is enough to check this on left-invariant vector fields.

$$\begin{aligned} d\omega_0(X^L,Y^L) &= \tfrac{1}{2}\big(X^L(\cancel{\omega_0(Y^L)}) - Y^L(\cancel{\omega_0(X^L)}) - \omega_0([X^L,Y^L])\big) \\ &= -\frac{1}{2}\omega_0([X^L,Y^L]) \\ &= -\frac{1}{2}[\omega_0,\omega_0](X^L,Y^L) \end{aligned}$$

$\rightsquigarrow \Big(\omega_0\wedge\omega_0(X^L,Y^L) = \frac{1}{2}(X^L_e\otimes Y^L_e - Y^L_e\otimes X^L_e)$

$\Rightarrow [\omega_0,\omega_0](X^L,Y^L) = \frac{1}{2}[X^L_e,Y^L_e] - \frac{1}{2}[Y^L_e,X^L_e]$

$$\rightsquigarrow \quad d\omega_{MC} = \pi_2^* d\omega_0 = \pi_2^* \frac{-1}{2}[\omega_0,\omega_0]$$
$$= -\frac{1}{2}[\pi_2^*\omega_0,\pi_2^*\omega_0] = -\frac{1}{2}[\omega_{MC},\omega_{MC}]$$

$\Rightarrow$ The curvature of $M\times G$ is 0!

<u>Definition</u> → A connection A in a principal G-bundle is called <u>flat</u> if the curvature form vanishes.

<u>3. Symplectic Structure on the space of connections</u>

$\mathcal{A}$ = Set of G-connections on P.

Let $A, A' \in \mathcal{A}$. Note that

$$\Rightarrow (A-A')_p(d_e\psi_p(\eta)) = \eta - \eta = 0$$

i.e. $A - A'$ is an "Horizontal" 1-form which is G-invariant in the sense of ii) of our definition of connection. We denote the set of such forms by $(\Omega^1_h(P)\otimes\mathfrak{g})^G$.

Lemma: $(\Omega^n_h(P)\otimes \mathfrak{g})^G \cong \Omega^n(X, \mathrm{ad}\, P)$ where $\mathrm{ad}\,P$ is the vector bundle, with fibre $\mathfrak{g}$ given by the fibered product $P \times_{Ad} \mathfrak{g}$

$\rightsquigarrow$ $\mathcal{A}$ is an infinite dimensional affine space modelled on the vector space

$$(\Omega^1_h(P)\otimes \mathfrak{g})^G \cong \Omega^1(X, \mathrm{ad}\, P)$$

Let us now describe the symplectic structure on $\mathcal{A}$...

A symplectic structure on an infinite dimensional manifold $\mathcal{A}$ is a closed 2-form ω on $\mathcal{A}$ + non-degenerate i.e. $\omega_A : T_A\mathcal{A} \to T_A^*\mathcal{A}$ is injective for all $A \in \mathcal{A}$.

Recall the Killing form $K(-,-): \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$
$(X^L, Y^L) \mapsto tr(ad(X^L)\circ ad(Y^L))$ — If G semisimple, otherwise take any G-invariant product on $\mathfrak{g}$

This, combined with the wedge product of forms gives a map:

$$K(-\wedge -): (\Omega^1_h(P)\otimes\mathfrak{g})^G \times (\Omega^1_h(P)\otimes \mathfrak{g})^G \longrightarrow \Omega^2_h(P)^G \overset{!!!}{\cong} \Omega^2(X)$$

We define the symplectic form $\omega_{\mathcal{A}}$ via the integration map:

$$\omega_{\mathcal{A}}(\alpha,\beta) = \int_X K(\alpha\wedge\beta)$$

4. The Gauge group action

Definition $\rightarrow$ The automorphism group

$$\mathcal{G}(P) = \{\varphi: P \to P \mid \pi\circ\varphi = \pi,\ \varphi(pg) = \varphi(p)g\}$$

is called the gauge group. $\Rightarrow$ • $\mathcal{G}(P)$ acts on $\mathcal{A}$ by pulling back a connection along a gauge equivalence: $\varphi\cdot A = (\varphi^{-1})^* A$ + define gauge equivalent connections

Furthermore, we can identify

$$\mathcal{G}(P) = \{f: P\to G \mid f(pg) = g^{-1} f(p) g\}$$
$$\cong \Gamma(\mathrm{Ad}\, P)$$

for $\mathrm{Ad}\,P = P\times_G G$ where G acts by conjugation on the second component.

This is an infinite Lie group whose Lie algebra can be identified with

$$\mathrm{Lie}(\mathcal{G}(P)) = \{\tilde{f}: P \to \mathfrak{g} \mid \tilde{f}(pg) = Ad(g^{-1})\tilde{f}(p)\} \cong \Gamma(\mathrm{ad}\, P)$$
$$\cong (\Omega^0_h(P)\otimes\mathfrak{g})^G$$

Integration gives a non-degenerate bilinear pairing

$$(\Omega_h^0(P)\otimes\mathfrak{g})^G \times (\Omega_h^2(P)\otimes\mathfrak{g})^G \longrightarrow \mathbb{R}$$
$$(h, F) \longmapsto \int_M K(h\wedge F)$$

So that we can identify $\mathrm{Lie}(\mathcal{G}(P))^* \cong (\Omega_h^2(P)\otimes\mathfrak{g})^G$.

⇝ The map $\mu: \mathcal{A} \longrightarrow \mathrm{Lie}(\mathcal{G}(P))^* \cong (\Omega_h^2(P)\otimes\mathfrak{g})^G$ → In fact $\mu = -d_A$.
$A \longmapsto -F_A$

is a moment map for the action of the Gauge group on $\mathcal{A}$. We need to check 3 things:

$h_A: TP \to TP$ onto the horizontal

→ Composition of projection with (exterior derivative) defined previously.

i) μ is well defined (Explain that $F_A = d_A(A)$)

ii) μ is $\mathcal{G}(P)$-equivariant

iii) $d_A\mu^\phi(\beta) = \omega_A(\phi^\#, \beta)$ for $A\in\mathcal{A}$, $\beta\in T_A\mathcal{A}\cong(\Omega_h^1 P\otimes\mathfrak{g})^G$ and $\phi\in \mathrm{Lie}\,\mathcal{G}(P)\cong(\Omega_h^0(P)\otimes\mathfrak{g})^G$.

Let us first compute the infinitesimal gauge action:

$$\phi^\#_A = \frac{d}{dt}\Big|_{t=0}(-\exp t\phi\cdot A) = -d_A\phi \in T_A\mathcal{A}$$

$$\Rightarrow d_A\mu^\phi(\beta) = \frac{d}{dt}\Big|_{t=0}\langle\mu(A+t\beta), \phi\rangle = \frac{d}{dt}\Big|_{t=0}\langle -F_A - t d_A\beta, \phi\rangle$$
$$= -\int_X K(d_A\beta\wedge\phi)$$
$$= -\int_X K(\beta\wedge d_A\phi)$$
$$= \int_X K(\phi^\#_A\wedge\beta) = \omega_A(\phi^\#_A, \beta)$$

$d_A K(\alpha\wedge\beta)$
$= K(d_A\alpha\wedge\beta) + K(\alpha\wedge d_A\beta)$

⇝ The moduli space of gauge equivalence classes of flat G-connections on P can be then constructed as

$$\mathcal{M}_F(X,G) := \mu^{-1}(0)/\mathcal{G}(P)$$

This can be shown to be a finite-dimensional symplectic orbifold.